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CALDERÓN-ZYGMUND OPERATORS ASSOCIATED WITH SCHRÖDINGER OPERATOR AND THEIR COMMUTATORS ON VANISHING GENERALIZED MORREY SPACES

A. AKBULUT[1], R. GULIYEV[2], I. EKINCIOGLU[2,3]

Abstract. We establish the boundedness of Calderón-Zygmund operators associated with Schrödinger operator and their commutators on vanishing generalized Morrey spaces.

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1. Introduction and results

The boundedness of the commutators of singular integrals plays an important role in harmonic analysis. Commutator integrals arise naturally when one tries to construct a calculus of singular integral operators to handle differential equations with nonsmooth coefficients.

Recently, the boundedness singular integrals and their commutators with BMO functions on Schrödinger operators settings have been received a great deal of attention. See for example [5, 6, 39, 41] and the references therein. In [4, 10, 34, 36, 37, 38], the authors proved the boundedness of singular integrals related to Schrödinger operators on $\mathbb{R}^n$ with certain potentials, and the boundedness of their commutators with BMO functions is studied.

The classical Morrey spaces were originally introduced by Morrey in [24] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [9, 12, 24, 30]. The classical version of Morrey spaces is equipped with the norm

$$\|f\|_{M_{p,\lambda}} := \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))},$$

where $0 \le \lambda \le n$ and $1 \le p < \infty$. Moreover, various Morrey spaces are defined in the process of study. V. Guliyev, Mizuhara and Nakai [14, 25, 26] introduced generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$ (see, also [1, 13, 15, 18, 20, 33]).

Let us consider the Schrödinger differential operator

$$L = -\Delta + V(x) \text{ on } \mathbb{R}^n, \ n \ge 3,$$

where $V(x)$ is a nonnegative potential belonging to the reverse Hölder class RH_q for $q \ge n/2$.

[1]Department of Mathematics, Kirsehir Ahi Evran University, Turkey
[2]Department of Mathematics, Kutahya Dumlupinar University, Turkey
[3]Department of Mathematics, Istanbul Medeniyet University, Turkey
e-mail: ramin@guliyev.com, ismail.ekincioglu@dpu.edu.tr
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A nonnegative locally L_q integrable function $V(x)$ on $\mathbb{R}^n$ is said to belong to RH_q, $1 < q \leq \infty$ if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B(x,r)|}\int\limits_{B(x,r)} V^q(y)dy\right)^{1/q} \leq \left(\frac{C}{|B(x,r)|}\int\limits_{B(x,r)} V(y)dy\right),$$

holds for every $x \in \mathbb{R}^n$ and $0 < r < \infty$, where $B(x,r)$ denotes the ball centered at x with radius r. In particular, if V is a nonnegative polynomial, then $V \in RH_\infty$.

For $x \in \mathbb{R}^n$, the function $\rho(x)$ is defined by

$$\rho(x) := \frac{1}{m_V(x)} = \sup_{r>0}\left\{ r : \frac{1}{r^{n-2}}\int\limits_{B(x,r)} V(y)dy \leq 1\right\}.$$

Obviously, $0 < m_V(x) < \infty$ if $V \neq 0$. In particular, $m_V(x) = 1$ with $V = 1$ and $m_V(x) \approx 1 + |x|$ with $V(x) = |x|^2$.

According to [4], the new BMO space $BMO_\theta(\rho)$ with $\theta \geq 0$ is defined as a set of all locally integrable functions b such that

$$\frac{1}{|B(x,r)|}\int\limits_{B(x,r)} |b(y) - b_B|dy \leq C\Big(1 + \frac{r}{\rho(x)}\Big)^{\theta},$$

for all $x \in \mathbb{R}^n$ and $r > 0$, where $b_B = \frac{1}{|B|}\int\limits_B b(y)dy$. A norm for $b \in BMO_\theta(\rho)$, denoted by $[b]_\theta$, is given by the infimum of the constants in the inequalities above. Clearly, $BMO \subset BMO_\theta(\rho)$.

Let $\theta > 0$ and $0 < \nu < 1$, in view of [23], the Campanato class, associated with Schrödinger operator $\Lambda_\nu^\theta(\rho)$ consists of the locally integrable functions b such that

$$\frac{1}{|B(x,r)|^{1+\nu/n}}\int\limits_{B(x,r)} |b(y) - b_B|dy \leq C\Big(1 + \frac{r}{\rho(x)}\Big)^{\theta}, \tag{1}$$

for all $x \in \mathbb{R}^n$ and $r > 0$. A seminorm of $b \in \Lambda_\nu^\theta(\rho)$, denoted by $[b]_\beta^\theta$, is given by the infimum of the constants in the inequality above.

Note that if $\theta = 0$, $\Lambda_\nu^\theta(\rho)$ is the classical Campanato space; if $\nu = 0$, $\Lambda_\nu^\theta(\rho)$ is exactly the space $BMO_\theta(\rho)$ introduced in [4].

For brevity, in the sequel we use the notations

$$\mathfrak{A}_{p,\varphi}^{\alpha,V}(f; x, r) := \Big(1 + \frac{r}{\rho(x)}\Big)^{\alpha} r^{-n/p}\, \varphi(x,r)^{-1} \|f\|_{L_p(B(x,r))},$$

and

$$\mathfrak{A}_{p,\varphi}^{W,\alpha,V}(f; x, r) := \Big(1 + \frac{r}{\rho(x)}\Big)^{\alpha} r^{-n/p}\, \varphi(x,r)^{-1} \|f\|_{WL_p(B(x,r))}.$$

We now present the definition of generalized Morrey spaces (including weak version) associated with Schrödinger operator, which introduced by V. Guliyev in [17], see also [19].

Definition 1.1. *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, $1 \leq p < \infty$, $\alpha \geq 0$, and $V \in RH_q$, $q > 1$. We denote by $M_{p,\varphi}^{\alpha,V} = M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ the generalized Morrey space associated with Schrödinger operator, the space of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{M_{p,\varphi}^{\alpha,V}} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{A}_{p,\varphi}^{\alpha,V}(f; x, r).$$

Also, $WM^{\alpha,V}_{p,\varphi} = WM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ *we denote the weak generalized Morrey space associated with Schrödinger operator, the space of all functions* $f \in WL^{loc}_p(\mathbb{R}^n)$ *with*

$$\|f\|_{WM^{\alpha,V}_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{A}^{W,\alpha,V}_{p,\varphi}(f;x,r) < \infty.$$

Remark 1.1. (i) *When* $\alpha = 0$, *and* $\varphi(x,r) = r^{(\lambda-n)/p}$, $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ *is the classical Morrey space* $L_{p,\lambda}(\mathbb{R}^n)$ *introduced by Morrey in* [24]*;*

(ii) *When* $\varphi(x,r) = r^{(\lambda-n)/p}$, $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ *is the Morrey space associated with Schrödinger operator* $L^{\alpha,V}_{p,\lambda}(\mathbb{R}^n)$ *studied by Tang and Dong in* [39]*;*

(iii) *The generalized Morrey space associated with Schrödinger operator* $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ *was introduced and studied by V. Guliyev in* [17].

Definition 1.2. *The vanishing generalized Morrey space associated with Schrödinger operator* $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ *is defined as the spaces of functions* $f \in M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ *such that*

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}^{\alpha,V}_{p,\varphi}(f;x,r) = 0. \tag{2}$$

The vanishing weak generalized Morrey space associated with Schrödinger operator $VWM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ *is defined as the spaces of functions* $f \in WM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ *such that*

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}^{W,\alpha,V}_{p,\varphi}(f;x,r) = 0.$$

The vanishing spaces $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ and $VWM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ are Banach spaces with respect to the norm

$$\|f\|_{VM^{\alpha,V}_{p,\varphi}} \equiv \|f\|_{M^{\alpha,V}_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{A}^{\alpha,V}_{p,\varphi}(f;x,r),$$

$$\|f\|_{VWM^{\alpha,V}_{p,\varphi}} \equiv \|f\|_{WM^{\alpha,V}_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{A}^{\alpha,V}_{W,p,\varphi}(f;x,r),$$

respectively.

In the case $\alpha = 0$, and $\varphi(x,r) = r^{(\lambda-n)/p}$ $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the vanishing Morrey space $VM_{p,\lambda}$ introduced in [40], where applications to PDE were considered.

We refer to [7, 31, 32] for some properties of vanishing generalized Morrey spaces.

From [34, 41], we know some Schrödinger type operators, such as $\nabla(-\Delta+V)^{-1}\nabla$ with $V \in B_n$, $\nabla(-\Delta+V)^{-1/2}$ with $V \in B_n$, $(-\Delta+V)^{-1/2}\nabla$ with $V \in B_n$, $(-\Delta+V)^{i\gamma}$ with $\gamma \in \mathbb{R}$ and $V \in B_{n/2}$, and $\nabla^2(-\Delta+V)^{-1}$ with V is a nonnegative polynomial, are standard Calderón-Zygmund operators, see [35]. In particular, the kernels K of operators above all satisfy

$$|K(x,y)| \le \frac{C_k}{\big(1+\frac{|x-y|}{\rho(x)}\big)^N} \frac{1}{|x-y|^n},$$

for any $N \in \mathbb{N}$. Hence, in the rest of this paper, we always assume that T denotes the above operators.

Let T be the classical singular integral operator, the commutator $[b,T]$ generated by T and a suitable function b is given by

$$[b,T](f)(x) = T\big((b(x)-b)f\big)(x) = b(x)T(f)(x) - T(bf)(x). \tag{3}$$

A well known result due to Coifman, Rochberg and Weiss [11] states that $[b,T]$ is bounded on $L^p(\mathbb{R}^n)$ for $1<p<\infty$ when $b \in BMO(\mathbb{R}^n)$. They also gave a characterization of $BMO(\mathbb{R}^n)$ in virtue of the L^p-boundedness of the above commutator. In 1978, Janson [21] gave some characterizations of Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ via commutator $[b,T]$ and proved that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ $(0<\beta<1)$ if and only if $[b,T]$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ where $1<p<n/\beta$ and $1/p - 1/q = \beta/n$.

The fractional integral associated with L is defined by

$$\mathcal{I}_\beta^L f(x) = L^{-\beta/2} f(x) = \int_0^\infty e^{-tL}(f)(x)\, t^{\beta/2-1} dt,$$

for $0 < \beta < n$. Note that, if $L = -\triangle$ is the Laplacian on $\mathbb{R}^n$, then $\mathcal{I}_\beta^L$ is the Riesz potential I_β, that is

$$I_\beta f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^n} dy.$$

When $b \in BMO$, Chanillo proved in [8] that $[b, I_\beta]$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ with $1/q = 1/p - \beta/n$, $1 < p < n/\beta$. When b belongs to the Campanato space Λ_ν, $0 < \nu < 1$, Paluszynski in [29] showed that $[b, I_\beta]$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ with $1/q = 1/p - (\beta + \nu)/n$, $1 < p < n/(\beta + \nu)$. When $b \in BMO_\theta(\rho)$, Bui in [6] obtained the boundedness of $[b, \mathcal{I}_\beta^L]$ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ with $1/q = 1/p - \beta/n$, $1 < p < n/\beta$.

It is well known that the boundedness of the standard Calderón-Zygmund operators and their commutators have been established on the class generalized Morrey spaces (see [15, 25, 26]). Hence, it will be an interesting question whether we can establish the boundedness of Schrödinger type operators on the vanishing generalized Morrey spaces related to certain nonnegative potentials (see [10, 17, 19, 22, 27, 28, 39]). The main purpose of this paper is to answer the above question. More precisely, we obtain the following results.

Theorem 1.1. *Let $V \in B_{n/2}$, $\alpha \geq 0$, $1 \leq p < \infty$ and $\varphi_1, \varphi_2 \in \Omega_{p,1}^{\alpha,V}$ satisfies the conditions*

$$c_\delta := \int_\delta^\infty \sup_{x \in \mathbb{R}^n} \varphi_1(x,t) \frac{dt}{t} < \infty, \tag{4}$$

for every $\delta > 0$, and

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{p}}} \frac{dt}{t} \leq C \varphi_2(x,r), \tag{5}$$

where C does not depend on x and r. Then the operator T is bounded on $VM_{p,\varphi_1}^{\alpha,V}$ to $VM_{p,\varphi_2}^{\alpha,V}$ for $p > 1$ and from $VM_{1,\varphi_1}^{\alpha,V}$ to $WVM_{1,\varphi_2}^{\alpha,V}$. Moreover, for $p > 1$

$$\|Tf\|_{VM_{p,\varphi_2}^{\alpha,V}} \leq C \|f\|_{VM_{p,\varphi_1}^{\alpha,V}},$$

and for $p = 1$

$$\|Tf\|_{WVM_{1,\varphi_2}^{\alpha,V}} \leq C \|f\|_{VM_{1,\varphi_1}^{\alpha,V}},$$

where C does not depend on f.

Theorem 1.2. *Let $V \in B_{n/2}$, $b \in BMO_\theta(\rho)$, $\alpha \geq 0$, $1 \leq p < \infty$ and $\varphi_1, \varphi_2 \in \Omega_{p,1}^{\alpha,V}$ satisfies the conditions*

$$\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{p}}} \frac{dt}{t} \leq C \varphi_2(x,r), \tag{6}$$

where C does not depend on x and r,

$$\lim_{r \to 0} \frac{\ln \frac{1}{r}}{\inf_{x \in \mathbb{R}^n} \varphi_2(x,r)} = 0 \tag{7}$$

and

$$c_\delta := \int_\delta^\infty \Big(1 + |\ln t|\Big) \sup_{x\in\mathbb{R}^n} \varphi_1(x,t)\frac{dt}{t} < \infty, \tag{8}$$

for every $\delta > 0$. If $b \in BMO_\theta(\rho)$, then the operator $[b,T]$ is bounded from $VM^{\alpha,V}_{p,\varphi_1}$ to $VM^{\alpha,V}_{p,\varphi_2}$ for $p > 1$ and from $VM^{\alpha,V}_{\Phi,\varphi_1}$ to $WVM^{\alpha,V}_{1,\varphi_2}$. Moreover, for $p > 1$

$$\|[b,T](f)\|_{VM^{\alpha,V}_{p,\varphi_2}} \le C[b]_\theta \|f\|_{VM^{\alpha,V}_{p,\varphi_1}},$$

and

$$\|[b,T](f)\|_{WVM^{\alpha,V}_{1,\varphi_2}} \le C[b]_\theta \|f\|_{VM^{\alpha,V}_{\Phi,\varphi_1}},$$

where $\Phi(t) = t\ln(e+t)$, $\|f\|_{VM^{\alpha,V}_{\Phi,\varphi}} = \|\Phi(|f|)\|_{VM^{\alpha,V}_{1,\varphi}}$, and C does not depend on f.

Theorem 1.3. *Let $V \in RH_{n/2}$, $0 < \nu < 1$, $b \in \Lambda^\theta_\nu(\rho)$, $1 < p < n/\nu$, $1/q = 1/p - \nu/n$, and $\varphi_1 \in \Omega^{\alpha,V}_{p,1}$, $\varphi_2 \in \Omega^{\alpha,V}_{q,1}$ satisfies the conditions (4) and*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t} \le C\varphi_2(x,r), \tag{9}$$

where C does not depend on $x \in \mathbb{R}^n$ and $r > 0$. Then the operator $[b,T]$ is bounded from $VM^{\alpha,V}_{p,\varphi_1}$ to $VM^{\alpha,V}_{q,\varphi_2}$ for $p > 1$ and from $VM^{\alpha,V}_{1,\varphi_1}$ to $VWM^{\alpha,V}_{\frac{n}{n-\nu},\varphi_2}$. Moreover, for $p > 1$

$$\|[b,T](f)\|_{VM^{\alpha,V}_{q,\varphi_2}} \le C\|f\|_{VM^{\alpha,V}_{p,\varphi_1}},$$

and for $p = 1$

$$\|[b,T](f)\|_{WVM^{\alpha,V}_{\frac{n}{n-\nu},\varphi_2}} \le C\|f\|_{VM^{\alpha,V}_{1,\varphi_1}},$$

where C does not depend on f.

Remark 1.2. *Note that, Theorems 1.1 and 1.2 in the case of $V \equiv 0$ was proved in* [16].

In this paper, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A \le CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2. Some Preliminaries

We would like to recall the important properties concerning the function $\rho(x)$.

Lemma 2.1. [34] *Let $V \in RH_{n/2}$. For the associated function ρ there exist C and $k_0 \ge 1$ such that*

$$C^{-1}\rho(x)\Big(1 + \frac{|x-y|}{\rho(x)}\Big)^{-k_0} \le \rho(y) \le C\rho(x)\Big(1 + \frac{|x-y|}{\rho(x)}\Big)^{\frac{k_0}{1+k_0}}, \tag{10}$$

for all $x, y \in \mathbb{R}^n$.

Lemma 2.2. [2] *Suppose $x \in B(x_0, r)$. Then for $k \in N$ we have*

$$\frac{1}{\Big(1 + \frac{2^k r}{\rho(x)}\Big)^N} \lesssim \frac{1}{\Big(1 + \frac{2^k r}{\rho(x_0)}\Big)^{N/(k_0+1)}}.$$

We give some inequalities about the new BMO space $BMO_\theta(\rho)$.

Lemma 2.3. [4] *Let* $1 \le s < \infty$. *If* $b \in BMO_\theta(\rho)$, *then*

$$\Big(\frac{1}{|B|}\int\limits_B |b(y)-b_B|^s dy\Big)^{1/s} \le [b]_\theta \Big(1+\frac{r}{\rho(x)}\Big)^{\theta'},$$

for all $B = B(x,r)$, *with* $x \in \mathbb{R}^n$ *and* $r > 0$, *where* $\theta' = (k_0+1)\theta$ *and* k_0 *is the constant appearing in* (10).

Lemma 2.4. [4] *Let* $1 \le s < \infty$, $b \in BMO_\theta(\rho)$, *and* $B = B(x,r)$. *Then*

$$\Big(\frac{1}{|2^kB|}\int\limits_{2^kB} |b(y)-b_B|^s dy\Big)^{1/s} \le [b]_\theta k \Big(1+\frac{2^k r}{\rho(x)}\Big)^{\theta'},$$

for all $k \in \mathbb{N}$, *with* θ' *as in Lemma 2.3.*

The Lipschitz space, associated with Schrödinger operator (see [23]) which consists of the functions f satisfying

$$\|f\|_{\mathrm{Lip}_\nu^\theta(\rho)} := \sup_{x\in\mathbb{R}^n,\, r>0} \frac{|f(x)-f(y)|}{|x-y|^\nu\big(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\big)^\theta} < \infty.$$

It is easy to see that this space is exactly the Lipschitz space when $\theta = 0$.

Note that if $\theta = 0$ in (1), $\Lambda_\nu^\theta(\rho)$ is exactly the classical Campanato space; if $\nu = 0$, $\Lambda_\nu^\theta(\rho)$ is exactly the space $BMO_\theta(\rho)$; if $\theta = 0$ and $\nu = 0$, it is exactly the John-Nirenberg space BMO.

The following relation between $\mathrm{Lip}_\nu^\theta(\rho)$ and $\Lambda_\nu^\theta(\rho)$ were proved in [23, Theorem 5].

Lemma 2.5. [23] *Let* $\theta > 0$ *and* $0 < \nu < 1$. *Then following embedding is valid*

$$\Lambda_\nu^\theta(\rho) \subseteq \mathrm{Lip}_\nu^\theta(\rho) \subseteq \Lambda_\nu^{(k_0+1)\theta}(\rho),$$

where k_0 *is the constant appearing in Lemma 2.1.*

We give some inequalities about the Campanato space, associated with Schrödinger operator $\Lambda_\nu^\theta(\rho)$.

Lemma 2.6. [23] *Let* $\theta > 0$ *and* $1 \le s < \infty$. *If* $b \in \Lambda_\nu^\theta(\rho)$, *then there exists a positive constant* C *such that*

$$\Big(\frac{1}{|B|}\int\limits_B |b(y)-b_B|^s dy\Big)^{1/s} \le C[b]_\nu^\theta r^\nu \Big(1+\frac{r}{\rho(x)}\Big)^{\theta'},$$

for all $B = B(x,r)$, *with* $x \in \mathbb{R}^n$ *and* $r > 0$, *where* $\theta' = (k_0+1)\theta$ *and* k_0 *is the constant appearing in* (10).

Let K_β be the kernel of $\mathcal{I}_\beta^L$. The following result give the estimate on the kernel $K_\beta(x,y)$.

Lemma 2.7. [6] *If* $V \in RH_{n/2}$, *then for every* N, *there exists a constant* C *such that*

$$|K_\beta(x,y)| \le \frac{C}{\Big(1+\frac{|x-y|}{\rho(x)}\Big)^N}\frac{1}{|x-y|^{n-\beta}}. \tag{11}$$

It is natural, first of all, to find conditions ensuring that the spaces $M_{p,\varphi}^{\alpha,V}$ and $VM_{p,\varphi}^{\alpha,V}$ are nontrivial, that is consist not only of functions equivalent to 0 on $\mathbb{R}^n$.

Lemma 2.8. [2] *Let* $\varphi(x,r)$ *be a positive measurable function on* $\mathbb{R}^n \times (0,\infty)$, $1 \le p < \infty$, $\alpha \ge 0$, *and* $V \in RH_q$, $q \ge 1$.

(i) *If*

$$\sup_{t<r<\infty}\Big(1+\frac{r}{\rho(x)}\Big)^{\alpha}\frac{r^{-\frac{n}{p}}}{\varphi(x,r)}=\infty \quad \text{for some } t>0 \ \text{ and for all } x\in\mathbb{R}^n,$$

then $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)=\Theta$.

(ii) *If*

$$\sup_{0<r<\tau}\Big(1+\frac{r}{\rho(x)}\Big)^{\alpha}\varphi(x,r)^{-1}=\infty \quad \text{for some } \tau>0 \ \text{ and for all } x\in\mathbb{R}^n,$$

then $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)=\Theta$.

Remark 2.1. *We denote by* $\Omega^{\alpha,V}_{p}$ *the sets of all positive measurable functions* φ *on* $\mathbb{R}^n\times(0,\infty)$ *such that for all* $t>0$,

$$\sup_{x\in\mathbb{R}^n}\Big\|\Big(1+\frac{r}{\rho(x)}\Big)^{\alpha}\frac{r^{-\frac{n}{p}}}{\varphi(x,r)}\Big\|_{L_\infty(t,\infty)}<\infty, \ \text{and} \ \sup_{x\in\mathbb{R}^n}\Big\|\Big(1+\frac{r}{\rho(x)}\Big)^{\alpha}\varphi(x,r)^{-1}\Big\|_{L_\infty(0,t)}<\infty.$$

For the non-triviality of the space $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ we always assume that $\varphi\in\Omega^{\alpha,V}_{p}$.

Remark 2.2. [3] *We denote by* $\Omega^{\alpha,V}_{p,1}$ *the sets of all positive measurable functions* φ *on* $\mathbb{R}^n\times(0,\infty)$ *such that*

$$\inf_{x\in\mathbb{R}^n}\inf_{r>\delta}\Big(1+\frac{r}{\rho(x)}\Big)^{-\alpha}\varphi(x,r)>0, \ \text{for some } \delta>0, \tag{12}$$

and

$$\lim_{r\to0}\Big(1+\frac{r}{\rho(x)}\Big)^{\alpha}\frac{r^{n/p}}{\varphi(x,r)}=0.$$

For the non-triviality of the space $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ we always assume that $\varphi\in\Omega^{\alpha,V}_{p,1}$.

Theorem 2.1. [18] *Let* $V\in B_{n/2}$, $\alpha\ge0$, $1\le p<\infty$ *and* $\varphi_1,\varphi_2\in\Omega^{\alpha,V}_{p}$ *satisfies the condition* (5). *Then the operator* T *is bounded on* $M^{\alpha,V}_{p,\varphi_1}$ *to* $M^{\alpha,V}_{p,\varphi_2}$ *for* $p>1$ *and from* $M^{\alpha,V}_{1,\varphi_1}$ *to* $WM^{\alpha,V}_{1,\varphi_2}$.

Theorem 2.2. [18] *Let* $V\in B_{n/2}$, $\alpha\ge0$, $1<p<\infty$ *and* $\varphi_1,\varphi_2\in\Omega^{\alpha,V}_{p}$ *satisfies the condition* (6). *If* $b\in BMO_\theta(\rho)$, *then the operator* $[b,T]$ *is bounded from* $M^{\alpha,V}_{p,\varphi_1}$ *to* $M^{\alpha,V}_{p,\varphi_2}$ *and from* $M^{\alpha,V}_{\Phi,\varphi_1}$ *to* $WM^{\alpha,V}_{1,\varphi_2}$.

Theorem 2.3. [3] *Let* $V\in B_{n/2}$, $\alpha\ge0$, $1<p<n/\nu$, $1/q=1/p-\nu/n$ *and* $\varphi_1\in\Omega^{\alpha,V}_{p}$, $\varphi_2\in\Omega^{\alpha,V}_{q}$ *satisfies the condition* (6). *If* $b\in\Lambda^{\theta}_{\nu}(\rho)$, *then the operator* $[b,T]$ *is bounded from* $M^{\alpha,V}_{p,\varphi_1}$ *to* $M^{\alpha,V}_{q,\varphi_2}$ *and from* $M^{\alpha,V}_{1,\varphi_1}$ *to* $WM^{\alpha,V}_{\frac{n}{n-\nu},\varphi_2}$.

3. Proof of Theorem 1.1

We first prove the following conclusions

Lemma 3.1. *Let* $0<\nu<1$, $0<\nu<1$ *and* $b\in\Lambda^{\theta}_{\nu}(\rho)$, *then the following pointwise estimate holds:*

$$\big|[b,T](f)(x)\big|\lesssim[b]^{\theta}_{\nu}\,I_\nu(|f|)(x).$$

Proof. Note that

$$[b,T](f)(x) = b(x)T(f)(x) - T(bf)(x) = \int_{\mathbb{R}^n} [b(x)-b(y)]\, K(x,y)\, f(y)dy.$$

If $b \in \Lambda_\nu^\theta(\rho)$, then from Lemma 2.7, we have

$$\begin{aligned}\big|[b,T](f)(x)\big| &\le \int_{\mathbb{R}^n} |b(x)-b(y)|\,|K(x,y)|\,|f(y)|dy \\ &\lesssim [b]_\nu^\theta \int_{\mathbb{R}^n} |x-y|^\nu |K(x,y)|\,|f(y)|dy = [b]_\nu^\theta\, I_\nu(|f|)(x).\end{aligned}$$

□

From Lemma 3.1, we get the following.

Corollary 3.1. *Suppose $V \in RH_{n/2}$ and $b \in \Lambda_\nu^\theta(\rho)$ with $0<\nu<1$. Let $1 \le p < q < \infty$ satisfy $1/q = 1/p - \nu/n$. Then for all f in $L_p(\mathbb{R}^n)$ we have*

$$\|[b,T](f)\|_{L_q(\mathbb{R}^n)} \lesssim \|f\|_{L_p(\mathbb{R}^n)},$$

when $p>1$, and also

$$\|[b,T](f)\|_{WL_{\frac{n}{n-\nu}}(\mathbb{R}^n)} \lesssim \|f\|_{L_1(\mathbb{R}^n)},$$

when $p=1$.

To prove Theorem 1.1, we used the following Guliyev type local estimate, see [18, Theorem 3.1.], see also [13].

Theorem 3.1. *Let $1<p<\infty$, then the inequality*

$$\|Tf\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t}, \tag{13}$$

holds for any ball $B(x_0,r)$ and for all $f \in L_p^{loc}(\mathbb{R}^n)$

Moreover, for $p=1$ the inequality

$$\|Tf\|_{WL_1(B(x_0,r))} \lesssim r^n \int_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^n} \frac{dt}{t},$$

holds for any ball $B(x_0,r)$ and for all $f \in L_1^{loc}(\mathbb{R}^n)$.

We are ready to start the proof of Theorem 1.1. The statement is derived from the estimate (13). The estimation of the norm of the operator, that is, the boundedness in the non-vanishing space, immediately follows from by Theorem 2.1. So, we only have to prove that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{p,\varphi_1}^{\alpha,V}(f;x,r) = 0 \;\Rightarrow\; \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{p,\varphi_2}^{\alpha,V}(Tf;x,r) = 0, \tag{14}$$

and

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{1,\varphi_1}^{\alpha,V}(f;x,r) = 0 \;\Rightarrow\; \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}_{1,\varphi_2}^{W,\alpha,V}(Tf;x,r) = 0. \tag{15}$$

To show that $\sup_{x\in\mathbb{R}^n} \left(1+\frac{r}{\rho(x)}\right)^\alpha \varphi_2(x,r)^{-1} r^{-n/p} \|Tf\|_{L_p(B(x,r))} < \varepsilon$ for small r, we split the right-hand side of (13):

$$\left(1+\frac{r}{\rho(x)}\right)^\alpha \varphi_2(x,r)^{-1} r^{-n/p} \|Tf\|_{L_p(B(x,r))} \le C[I_{\delta_0}(x,r) + J_{\delta_0}(x,r)], \tag{16}$$

where $\delta_0 > 0$ (we may take $\delta_0 > 1$), and

$$I_{\delta_0}(x,r) := \frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_2(x,r)} \int\limits_r^{\delta_0} t^{-\frac{n}{p}-1} \|f\|_{L_p(B(x,t))} dt,$$

and

$$J_{\delta_0}(x,r) := \frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_2(x,r)} \int\limits_{\delta_0}^{\infty} t^{-\frac{n}{p}-1} \|f\|_{L_p(B(x,t))} dt,$$

and it is supposed that $r < \delta_0$. We use the fact that $f \in VM^{\alpha,V}_{p,\varphi_1}(\mathbb{R}^n)$ and choose any fixed $\delta_0 > 0$ such that

$$\sup_{x\in\mathbb{R}^n} \left(1+\frac{t}{\rho(x)}\right)^{\alpha} \varphi_1(x,t)^{-1} t^{-n/p} \|f\|_{L_p(B(x,t))} < \frac{\varepsilon}{2CC_0},$$

where C and C_0 are constants from (5) and (22). This allows to estimate the first term uniformly in $r \in (0, \delta_0)$:

$$\sup_{x\in\mathbb{R}^n} C I_{\delta_0}(x,r) < \frac{\varepsilon}{2}, \ 0 < r < \delta_0.$$

The estimation of the second term now my be made already by the choice of r sufficiently small. Indeed, thanks to the condition (12) we have

$$J_{\delta_0}(x,r) \le c_{\delta_0} \ \frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_1(x,r)} \ \|f\|_{VM^{\alpha,V}_{p,\varphi_1}},$$

where c_{δ_0} is the constant from (2). Then, by (12) it suffices to choose r small enough such that

$$\sup_{x\in\mathbb{R}^n} \frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_2(x,r)} \le \frac{\varepsilon}{2c_{\delta_0}\|f\|_{VM^{\alpha,V}_{p,\varphi_1}}},$$

which completes the proof of (14).

The proof of (15) is similar to the proof of (14).

4. Proof of Theorem 1.2

Similar to the proof of Theorem 1.2, we used the following Guliyev type local estimate, see [18, Theorem 4.1.], see also [13].

Theorem 4.1. *Let $V \in RH_{n/2}$, $b \in BMO_\theta(\rho)$. If $1 < p < \infty$, then the inequality*

$$\|[b,T](f)\|_{L_p(B(x_0,r))} \lesssim [b]_\theta r^{\frac{n}{p}} \int\limits_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t}, \qquad (17)$$

holds for any ball $B(x_0,r)$ and for all $f \in L^{loc}_p(\mathbb{R}^n)$.

Moreover, the inequality

$$\|[b,T](f)\|_{WL_1(B(x_0,r))} \lesssim [b]_\theta\, r^n \int\limits_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) \frac{\|f\|_{L_\Phi(B(x_0,t))}}{t^n} \frac{dt}{t},$$

holds for any ball $B(x_0,r)$ and for all $f \in L^{loc}_1(\mathbb{R}^n)$, where $\Phi(t) = t\ln(e+t)$.

The norm inequality having already been provided by Theorem 2.2, we only have to prove the implication

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\Big(1+\frac{r}{\rho(x)}\Big)^{\alpha}\varphi_1(x,r)^{-1}r^{-n/p}\|f\|_{L_p(B(x,r))}=0$$
$$\Longrightarrow \lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\Big(1+\frac{r}{\rho(x)}\Big)^{\alpha}\varphi_2(x,r)^{-1}r^{-n/p}\|[b,T](f)\|_{L_p(B(x,r))}=0.$$

To check that

$$\sup_{x\in\mathbb{R}^n}\Big(1+\frac{r}{\rho(x)}\Big)^{\alpha}\varphi_2(x,r)^{-1}r^{-n/p}\|[b,T](f)\|_{L_p(B(x,r))}<\varepsilon \quad \text{for small } \ r,$$

we use the estimate (17):

$$\varphi_2(x,r)^{-1}r^{-n/p}\|[b,T](f)\|_{L_p(B(x,r))}\lesssim\frac{[b]_\theta}{\varphi_2(x,r)}\int_r^\infty\Big(1+\ln\frac{t}{r}\Big)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}}\frac{dt}{t}.$$

We take $r<\delta_0$, where δ_0 will be chosen small enough and split the integration:

$$\Big(1+\frac{r}{\rho(x)}\Big)^{\alpha}\varphi_2(x,r)^{-1}r^{-n/p}\|[b,T](f)\|_{L_p(B(x,r))}\le C[I_{\delta_0}(x,r)+J_{\delta_0}(x,r)], \tag{18}$$

where

$$I_{\delta_0}(x,r):=\frac{\big(1+\frac{r}{\rho(x)}\big)^{\alpha}}{\varphi_2(x,r)}\int_r^{\delta_0}\Big(1+\ln\frac{t}{r}\Big)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}}\frac{dt}{t}$$

and

$$J_{\delta_0}(x,r):=\frac{\big(1+\frac{r}{\rho(x)}\big)^{\alpha}}{\varphi_2(x,r)}\int_{\delta_0}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}}\frac{dt}{t}.$$

We choose a fixed $\delta_0>0$ such that

$$\sup_{x\in\mathbb{R}^n}\Big(1+\frac{r}{\rho(x)}\Big)^{\alpha}\varphi_1(x,r)^{-1}r^{-n/p}\|f\|_{L_p(B(x,r))}<\frac{\varepsilon}{2CC_0},\quad r\le\delta_0,$$

where C and C_0 are constants from (18) and (6), which yields the estimate of the first term uniform in $r\in(0,\delta_0):\ \sup\limits_{x\in\mathbb{R}^n}CI_{\delta_0}(x,r)<\frac{\varepsilon}{2},\ 0<r<\delta_0$.

For the second term, writing $1+\ln\frac{t}{r}\le 1+|\ln t|+\ln\frac{1}{r}$, we obtain

$$J_{\delta_0}(x,r)\le\frac{c_{\delta_0}+\widetilde{c_{\delta_0}}\ \ln\frac{1}{r}}{\varphi_2(x,r)}\|f\|_{M^{\alpha,V}_{p,\varphi_1}},$$

where c_{δ_0} is the constant from (8) with $\delta=\delta_0$ and $\widetilde{c_{\delta_0}}$ is a similar constant with omitted logarithmic factor in the integrand. Then, by (7) we can choose small r such that $\sup_{x\in\mathbb{R}^n}J_{\delta_0}(x,r)<\frac{\varepsilon}{2}$, which completes the proof.

5. Proof of Theorem 1.3

Similar to the proof of Theorem 1.3, we used the following Guliyev type local estimate, see [3, Theorem 5.]. In order to prove Theorem 1.3, we need the following

Theorem 5.1. *Suppose $V\in RH_{n/2}$, $b\in\Lambda^\theta_\nu(\rho)$, $\theta>0$, $0<\nu<1$. Let $1\le p<q<\infty$ satisfy $1/q=1/p-\nu/n$ then the inequality*

$$\|[b,T](f)\|_{L_q(B(x_0,r))}\lesssim\|I_\nu(|f|)\|_{L_q(B(x_0,r))}\lesssim r^{\frac{n}{q}}\int_{2r}^{\infty}\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}, \tag{19}$$

holds for any $f \in L_p^{loc}(\mathbb{R}^n)$. *Moreover, for* $p=1$ *the inequality*

$$\|[b,T](f)\|_{WL_{\frac{n}{n-\nu}}(B(x_0,r))} \lesssim \|I_\nu(|f|)\|_{WL_{\frac{n}{n-\nu}}(B(x_0,r))} \lesssim r^n \int\limits_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\nu}} \frac{dt}{t},$$

holds for any $f \in L_1^{loc}(\mathbb{R}^n)$.

We are ready to start the proof of Theorem 1.3. The statement is derived from the estimate (19). The estimation of the norm of the operator, that is, the boundedness in the non-vanishing space, immediately follows from by Theorem 2.3. So, we only have to prove that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}^{\alpha,V}_{p,\varphi_1}(f;x,r) = 0 \ \Rightarrow \ \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}^{\alpha,V}_{q,\varphi_2}([b,T](f);x,r) = 0, \tag{20}$$

and

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}^{\alpha,V}_{1,\varphi_1}(f;x,r) = 0 \ \Rightarrow \ \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}^{W,\alpha,V}_{1,\varphi_2}([b,T](f);x,r) = 0. \tag{21}$$

To show that $\sup\limits_{x\in\mathbb{R}^n} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \varphi_2(x,r)^{-1} r^{-n/p} \|[b,T](f)\|_{L_q(B(x,r))} < \varepsilon$ for small r, we split the right-hand side of (19):

$$\left(1+\frac{r}{\rho(x)}\right)^{\alpha} \varphi_2(x,r)^{-1} r^{-n/p} \|[b,T](f)\|_{L_q(B(x,r))} \le C\left[I_{\delta_0}(x,r) + J_{\delta_0}(x,r)\right], \tag{22}$$

where $\delta_0 > 0$ (we may take $\delta_0 > 1$), and

$$I_{\delta_0}(x,r) := \frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_2(x,r)} \int\limits_{r}^{\delta_0} t^{-\frac{n}{q}-1} \|f\|_{L_p(B(x,t))} dt,$$

and

$$J_{\delta_0}(x,r) := \frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_2(x,r)} \int\limits_{\delta_0}^{\infty} t^{-\frac{n}{q}-1} \|f\|_{L_p(B(x,t))} dt,$$

and it is supposed that $r < \delta_0$. We use the fact that $f \in VM^{\alpha,V}_{p,\varphi_1}(\mathbb{R}^n)$ and choose any fixed $\delta_0 > 0$ such that

$$\sup_{x\in\mathbb{R}^n} \left(1+\frac{t}{\rho(x)}\right)^{\alpha} \varphi_1(x,t)^{-1} t^{-n/p} \|f\|_{L_p(B(x,t))} < \frac{\varepsilon}{2CC_0},$$

where C and C_0 are constants from (9) and (22). This allows to estimate the first term uniformly in $r \in (0,\delta_0)$:

$$\sup_{x\in\mathbb{R}^n} C I_{\delta_0}(x,r) < \frac{\varepsilon}{2}, \ 0 < r < \delta_0.$$

The estimation of the second term now my be made already by the choice of r sufficiently small. Indeed, thanks to the condition (12) we have

$$J_{\delta_0}(x,r) \le c_{\delta_0} \frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_1(x,r)} \|f\|_{VM^{\alpha,V}_{p,\varphi_1}},$$

where c_{δ_0} is the constant from (2). Then, by (12) it suffices to choose r small enough such that

$$\sup_{x\in\mathbb{R}^n} \frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_2(x,r)} \le \frac{\varepsilon}{2c_{\delta_0}\|f\|_{VM^{\alpha,V}_{p,\varphi_1}}},$$

which completes the proof of (20).

The proof of (21) is similar to the proof of (20).

6. Conclusion

In this paper, we have studied the boundedness of Calderón-Zygmund operators T associated with Schrödinger operator and their commutators $[b, T]$ with $b \in BMO_\theta(\rho)$ or $b \in \Lambda_\nu^\theta(\rho)$ on vanishing generalized Morrey spaces $VM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ related to Schrödinger operator.

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Ali Akbulut - is a Professor at the Department of Mathematics of Kirsehir Ahi Evran University. He is graduated from the Department of Mathematics of Art and Science Faculty of Ankara University in 1994. He received the Ph.D. degree from Ankara University Graduate School of Natural and Applied Sciences in 2003. His research interests are theory of integral operators in Morrey-types paces, regularity of the solution of elliptic and parabolic type differential equations.

Ramin Guliyev - is a Ph.D. student at the Department of Mathematics of Dumlupinar University. He is graduated from faculty of Applied Mathematics and Cybernetics of the Baku State University in 2005. His research interests are harmonic analysis, differential equations.

Ismail Ekincioglu - is a Professor at the Department of Mathematics of Istanbul Medeniyet University. He is graduated from the Department of Mathematics of Art and Science Faculty of Ataturk University in 1987. He received the Ph.D. degree from Ankara University Graduate School of Natural and Applied Sciences in 1995. His research interests are harmonic analysis, differential equations.