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HIGHER ORDER RIESZ TRANSFORMS RELATED TO SCHRÖDINGER TYPE OPERATOR ON LOCAL GENERALIZED MORREY SPACES

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Abstract. In this paper, we study the boundedness of the higher order Riesz transforms $\mathcal{R}$, $\mathcal{R}^*$ and their commutators $[b,\mathcal{R}]$, $[b,\mathcal{R}^*]$ on local generalized Morrey spaces $LM_{p,\varphi}^{\alpha,V,\{x_0\}}$ and vanishing generalized Morrey spaces $VM_{p,\varphi}^{\alpha,V}$ related to Schrödinger type operator. We find the sufficient conditions on the pair (φ_1,φ_2) which ensures the boundedness of these operators from one local generalized Morrey space $LM_{p,\varphi}^{\alpha,V,\{x_0\}}$ to another $LM_{p,\varphi_2}^{\alpha,V,\{x_0\}}$ and from one vanishing generalized Morrey space $VM_{p,\varphi}^{\alpha,V}$ to another $VM_{p,\varphi_2}^{\alpha,V}$.

Keywords: Schrödinger type operator, higher order Riesz transform, commutator, BMO, local generalized Morrey space.

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1. Introduction and results

Let us consider the Schrödinger operator

$$\mathcal{L}_2 = (-\triangle)^2 + V^2(x) \text{ on } \mathbb{R}^n,\ n \geq 5,$$

where V is non-negative, $V \neq 0$, and belongs to a reverse Hölder class RH_q for some $q \geq n/2$. i.e., there exists a constant C such that

$$\Big(\frac{1}{|B|}\int\limits_B V(y)^q dy\Big)^{1/q} \leq \frac{C}{|B|}\int\limits_B V(y)dy$$

for every ball $B \subset \mathbb{R}^n$.

Obviously, $RH_{q_2} \subset RH_{q_1}$, if $q_2 > q_1$. But it is important that the class RH_q has a property of self improvement, that is, if $V \in RH_q$, then $V \in RH_{q+\epsilon}$ for some $\epsilon > 0$. We define the reverse Hölder index of V as $q_0 = \sup\{q : V \in RH_q\}$.

As in [26], for a given potential $V \in RH_q$ with $q > n/2$, we define the auxiliary function

$$\rho(x) = \sup\Big\{r > 0 : \frac{1}{r^{d-2}}\int\limits_{B(x,r)} V(y)dy \leq 1\Big\},\ x \in \mathbb{R}^n.$$

It is well known that $0 < \rho(x) < \infty$ for any $x \in \mathbb{R}^n$.

Obviously, $0 < m_V(x) < \infty$ if $V \neq 0$. In particular, $m_V(x) = 1$ with $V = 1$ and $m_V(x) \sim 1 + |x|$ with $V(x) = |x|^2$.

Note that if $P(x)$ is a polynomial and $\beta > 0$, it is easy to see that $V(x) = |P(x)|^\beta$ belongs to RH_{q_1} for $q_1 \geq n/2$ and there exists a constant C such that $V(x) \leq C m_V(x)^2$ (see [16, 19]).

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According to [4], the new BMO space $BMO_\theta(\rho)$ with $\theta \geq 0$ is defined as a set of all locally integrable functions b such that

$$\frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |b(y) - b_B| dy \leq C\Big(1 + \frac{r}{\rho(x)}\Big)^\theta$$

for all $x \in \mathbb{R}^n$ and $r > 0$, where $b_B = \frac{1}{|B|} \int\limits_B b(y)dy$. A norm for $b \in BMO_\theta(\rho)$, denoted by $[b]_\theta$, is given by the infimum of the constants in the inequalities above. Clearly, $BMO \subset BMO_\theta(\rho)$.

The classical Morrey spaces were originally introduced by Morrey in [20] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [6, 7, 10, 14, 15, 20, 23, 25]. The classical version of Morrey spaces is equipped with the norm

$$\|f\|_{L_{p,\lambda}} := \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))}, \tag{1}$$

where $0 \leq \lambda < n$ and $1 \leq p < \infty$. The generalized Morrey spaces are defined with r^λ replaced by a general non-negative function $\varphi(x,r)$ satisfying some assumptions (see, for example, [10, 18, 21] and etc).

The vanishing Morrey space $VL_{p,\lambda}$ in its classical version was introduced in [28], where applications to PDE were considered. We also refer to [5] and [22] for some properties of such spaces. This is a subspace of functions in $L_{p,\lambda}(\mathbb{R}^n)$, which satisfy the condition

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n,\, 0<t<r} t^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,t))} = 0.$$

We now present the definition of generalized Morrey spaces (including weak version) related to Schrödinger operator, which introduced by V. Guliyev in [12].

Definition 1.1. *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, $1 \leq p < \infty$, $\alpha \geq 0$, and $V \in RH_q$, $q \geq 1$. We denote by $M^{\alpha,V}_{p,\varphi} = M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space related to Schrödinger operator, the space of all functions $f \in L^{loc}_p(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{M^{\alpha,V}_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \Big(1 + \frac{r}{\rho(x)}\Big)^\alpha \varphi(x,r)^{-1} r^{-n/p} \|f\|_{L_p(B(x,r))}.$$

Also $WM^{\alpha,V}_{p,\varphi} = WM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space related to Schrödinger operator,the space of all functions $f \in WL^{loc}_p(\mathbb{R}^n)$ with

$$\|f\|_{WM^{\alpha,V}_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \Big(1 + \frac{r}{\rho(x)}\Big)^\alpha \varphi(x,r)^{-1} r^{-n/p} \|f\|_{WL_p(B(x,r))} < \infty.$$

Remark 1.1. (*i*) *When $\alpha = 0$, and $\varphi(x,r) = r^{(\lambda-n)/p}$, $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the classical Morrey space $L_{p,\lambda}(\mathbb{R}^n)$ introduced by Morrey in* [20]*;*

(*ii*) *When $\varphi(x,r) = r^{(\lambda-n)/p}$, $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the Morrey space related to Schrödinger operator $L^{\alpha,V}_{p,\lambda}(\mathbb{R}^n)$ studied by Tang and Dong in* [27]*;*

(*iii*) *When $\alpha = 0$, $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the generalized Morrey space $M_{p,\varphi}(\mathbb{R}^n)$ introduced by Mizuhara and Nakai in* [18, 21]*;*

(*iv*) *The generalized Morrey space related to Schrödinger operator $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ was introduced by Guliyev in* [12].

For brevity, in the sequel we use the notations

$$\mathfrak{A}^{\alpha,V}_{p,\varphi}(f;x,r) := \Big(1 + \frac{r}{\rho(x)}\Big)^\alpha r^{-n/p}\, \varphi(x,r)^{-1} \|f\|_{L_p(B(x,r))}$$

and

$$\mathfrak{A}^{W,\alpha,V}_{\Phi,\varphi}(f;x,r) := \Big(1 + \frac{r}{\rho(x)}\Big)^\alpha r^{-n/p}\, \varphi(x,r)^{-1} \|f\|_{WL_p(B(x,r))}.$$

Definition 1.2. *The vanishing generalized Morrey space related to Schrödinger operator $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ such that*

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}^{\alpha,V}_{p,\varphi}(f;x,r) = 0. \tag{2}$$

The vanishing weak generalized Morrey space related to Schrödinger operator $VWM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in WM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}^{W,\alpha,V}_{p,\varphi}(f;x,r) = 0.$$

The vanishing spaces $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ and $VWM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ are Banach spaces with respect to the norm

$$\|f\|_{VM^{\alpha,V}_{p,\varphi}} \equiv \|f\|_{M^{\alpha,V}_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{A}^{\alpha,V}_{p,\varphi}(f;x,r),$$
$$\|f\|_{VWM^{\alpha,V}_{p,\varphi}} \equiv \|f\|_{WM^{\alpha,V}_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{A}^{W,\alpha,V}_{p,\varphi}(f;x,r),$$

respectively.

Remark 1.2. (i) *When $\alpha = 0$, and $\varphi(x,r) = r^{(\lambda-n)/p}$, $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the vanishing Morrey space $VL_{p,\lambda}(\mathbb{R}^n)$ introduced by Vitanza in* [28]*;*

(ii) *When $\alpha = 0$, $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the vanishing generalized Morrey space $VM_{p,\varphi}(\mathbb{R}^n)$ studied in* [1, 24].

(iii) *The vanishing generalized Morrey space related to Schrödinger operator $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ were studied in* [2].

Definition 1.3. *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, $1 \le p < \infty$, $\alpha \ge 0$, and $V \in RH_q$, $q \ge 1$. For any fixed $x_0 \in \mathbb{R}^n$ we denote by $LM^{\alpha,V,\{x_0\}}_{p,\varphi} = LM^{\alpha,V,\{x_0\}}_{p,\varphi}(\mathbb{R}^n)$ the local generalized Morrey space related to Schrödinger operator, the space of all functions $f \in L^{loc}_p(\mathbb{R}^n)$ with finite norm*

$$\|f\|_{LM^{\alpha,V,\{x_0\}}_{p,\varphi}} = \sup_{r>0} \mathfrak{A}^{\alpha,V}_{p,\varphi}(f;x_0,r).$$

Also $WLM^{\alpha,V,\{x_0\}}_{p,\varphi} = WLM^{\alpha,V,\{x_0\}}_{p,\varphi}(\mathbb{R}^n)$ we denote the weak local generalized Morrey space related to Schrödinger operator,the space of all functions $f \in WL^{loc}_p(\mathbb{R}^n)$ with

$$\|f\|_{WLM^{\alpha,V,\{x_0\}}_{p,\varphi}} = \sup_{r>0} \mathfrak{A}^{W,\alpha,V}_{p,\varphi}(f;x_0,r) < \infty.$$

The local spaces $LM^{\alpha,V,\{x_0\}}_{p,\varphi}(\mathbb{R}^n)$ and $WLM^{\alpha,V,\{x_0\}}_{p,\varphi}(\mathbb{R}^n)$ are Banach spaces with respect to the norm

$$\|f\|_{LM^{\alpha,V,\{x_0\}}_{p,\varphi}} = \sup_{r>0} \mathfrak{A}^{\alpha,V}_{p,\varphi}(f;x_0,r),$$
$$\|f\|_{WLM^{\alpha,V,\{x_0\}}_{p,\varphi}} = \sup_{r>0} \mathfrak{A}^{W,\alpha,V}_{p,\varphi}(f;x_0,r),$$

respectively.

Remark 1.3. (i) *When $\alpha = 0$, and $\varphi(x,r) = r^{(\lambda-n)/p}$, $LM^{\alpha,V,\{x_0\}}_{p,\varphi}(\mathbb{R}^n)$ is the local (central) Morrey space $LM^{\{0\}}_{p,\lambda}(\mathbb{R}^n)$ studied in* [3]*;*

(ii) *When $\alpha = 0$, $LM^{\alpha,V,\{x_0\}}_{p,\varphi}(\mathbb{R}^n)$ is the local generalized Morrey space $VM^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n)$ were introduced by Guliyev in* [8]*, see also* [9, 11] *etc.*

It is natural, first of all, to find conditions ensuring that the spaces $LM^{\alpha,V,\{x_0\}}_{p,\varphi}$ and $M^{\alpha,V}_{p,\varphi}$ are nontrivial, that is consist not only of functions equivalent to 0 on $\mathbb{R}^n$.

Lemma 1.1. *Let $x_0 \in \mathbb{R}^n$, $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, $1 \le p < \infty$, $\alpha \ge 0$, and $V \in RH_q$, $q \ge 1$. If*

$$\sup_{t<r<\infty} \Big(1+\frac{r}{\rho(x_0)}\Big)^{\alpha} \frac{r^{-\frac{n}{p}}}{\varphi(x_0,r)} = \infty \quad \text{for some } t>0, \tag{3}$$

then $LM_{p,\varphi}^{\alpha,V,\{x_0\}}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Proof. Let (4) be satisfied and f be not equivalent to zero. Then $\|f\|_{L_p(B(x_0,t))} > 0$, hence

$$\begin{aligned}\|f\|_{LM_{p,\varphi}^{\alpha,V,\{x_0\}}} &\ge \sup_{t<r<\infty} \Big(1+\frac{r}{\rho(x_0)}\Big)^{\alpha} \varphi(x_0,r)^{-1} r^{-\frac{n}{p}} \|f\|_{L_p(B(x_0,r))} \\ &\ge \|f\|_{L_p(B(x_0,t))} \sup_{t<r<\infty} \Big(1+\frac{r}{\rho(x_0)}\Big)^{\alpha} \varphi(x_0,r)^{-1} r^{-\frac{n}{p}}.\end{aligned}$$

Therefore $\|f\|_{LM_{p,\varphi}^{\alpha,V,\{x_0\}}} = \infty$. □

Lemma 1.2. [2] *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, $1 \le p < \infty$, $\alpha \ge 0$, and $V \in RH_q$, $q \ge 1$.*

(i) If

$$\sup_{t<r<\infty} \Big(1+\frac{r}{\rho(x)}\Big)^{\alpha} \frac{r^{-\frac{n}{p}}}{\varphi(x,r)} = \infty \quad \text{for some } t>0 \ \text{ and for all } x \in \mathbb{R}^n, \tag{4}$$

then $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n) = \Theta$.

(ii) If

$$\sup_{0<r<\tau} \Big(1+\frac{r}{\rho(x)}\Big)^{\alpha} \varphi(x,r)^{-1} = \infty \quad \text{for some } \tau>0 \ \text{ and for all } x \in \mathbb{R}^n, \tag{5}$$

then $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n) = \Theta$.

Remark 1.4. *We denote by $\Omega_{p,loc}^{\alpha,V}$ the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0,\infty)$ such that for all $t>0$,*

$$\sup_{x\in\mathbb{R}^n} \Big\| \Big(1+\frac{r}{\rho(x)}\Big)^{\alpha} \frac{r^{-\frac{n}{p}}}{\varphi(x,r)} \Big\|_{L_\infty(t,\infty)} < \infty.$$

Moreover, we denote by $\Omega_p^{\alpha,V}$ the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0,\infty)$ such that for all $t>0$,

$$\sup_{x\in\mathbb{R}^n} \Big\| \Big(1+\frac{r}{\rho(x)}\Big)^{\alpha} \frac{r^{-\frac{n}{p}}}{\varphi(x,r)} \Big\|_{L_\infty(t,\infty)} < \infty, \text{ and } \sup_{x\in\mathbb{R}^n} \Big\| \Big(1+\frac{r}{\rho(x)}\Big)^{\alpha} \varphi(x,r)^{-1} \Big\|_{L_\infty(0,t)} < \infty.$$

Remark 1.5. *We denote by $\Omega_{p,1}^{\alpha,V}$ the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0,\infty)$ such that*

$$\inf_{x\in\mathbb{R}^n} \inf_{r>\delta} \Big(1+\frac{r}{\rho(x)}\Big)^{-\alpha} \varphi(x,r) > 0, \text{ for some } \delta>0, \tag{6}$$

and

$$\lim_{r\to 0} \Big(1+\frac{r}{\rho(x)}\Big)^{\alpha} \frac{r^{n/p}}{\varphi(x,r)} = 0. \tag{7}$$

For the non-triviality of the spaces $LM_{p,\varphi}^{\alpha,V,\{x_0\}}(\mathbb{R}^n)$, $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ and $VM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ we always assume that $\varphi \in \Omega_{p,loc}^{\alpha,V}$, $\varphi \in \Omega_p^{\alpha,V}$ and $\varphi \in \Omega_{p,1}^{\alpha,V}$, respectively.

The Riesz transform related to $\mathcal{L} = -\triangle + V$ is defined by $\mathcal{R}_1 = \nabla \mathcal{L}^{-1/2}$, and its dual is defined by $\mathcal{R}_1^* = \mathcal{L}^{-1/2}\nabla$. The L_p boundedness of $\mathcal{R}$ and $\mathcal{R}^*$ have been obtained in [26] by Shen. Let $b \in BMO_\theta(\rho)$, Bongioanni, Harboure and Salinas in [4] showed that the commutators $[b,\mathcal{R}_1]$ and $[b,\mathcal{R}_1^*]$ are also bounded on L_p. In [13], was proved that the operators $\mathcal{R}_1^*$ and $[b,\mathcal{R}_1^*]$ with

$b \in BMO_\theta(\rho)$ are bounded on $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ and $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$. In [2] showed that the Marcinkiewicz operators μ^L_j and their commutators $[b, \mu^L_j]$ with $b \in BMO_\theta(\rho)$ are bounded on $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ and $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$.

The higher order Riesz transform related to $\mathcal{L}_2$ is defined by $\mathcal{R} = \nabla^2\mathcal{L}_2^{-1/2}$, and its dual is defined by $\mathcal{R}^* = \mathcal{L}_2^{-1/2}\nabla^2$. The L_p boundedness of $\mathcal{R}$ and $\mathcal{R}^*$ have been obtained in [16] by Liu and Dong. Let $b \in BMO_\theta(\rho)$, Liu et al. in [17] showed that the commutators $[b, \mathcal{R}]$ and $[b, \mathcal{R}^*]$ are also bounded on L_p.

In this paper, we consider the boundedness of the operators $\mathcal{R}$ and $\mathcal{R}^*$ on local generalized Morrey space $LM^{\alpha,V,\{x_0\}}_{p,\varphi}(\mathbb{R}^n)$, generalized Morrey space $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ and vanishing generalized Morrey space $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ related to Schrödinger type operator. When b belongs to the new BMO function spaces $BMO_\theta(\rho)$, we also show that the commutator operators $[b, \mathcal{R}]$ and $[b, \mathcal{R}^*]$ are bounded on $LM^{\alpha,V,\{x_0\}}_{p,\varphi}(\mathbb{R}^n)$, $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ and $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$.

Our main results are as follows.

Theorem 1.1. *Let $x_0 \in \mathbb{R}^n$, $V \in RH_q$ with $n/2 \le q < n$, $\alpha \ge 0$, $1/p_0 = 2/q_0 - 2/n$, q_0 is the reverse Hölder index of V, and $\varphi_1, \varphi_2 \in \Omega^{\alpha,V}_{p,loc}$ satisfies the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0, s)s^{\frac{n}{p}}}{t^{\frac{n}{p}}} \frac{dt}{t} \le c_0\varphi_2(x_0, r), \tag{8}$$

where c_0 does not depend on r.

(i) If $p = 1$, then the operator $\mathcal{R}$ is bounded from $LM^{\alpha,V,\{x_0\}}_{1,\varphi_1}$ to $WLM^{\alpha,V,\{x_0\}}_{1,\varphi_2}$. Moreover, there exists a constant C such that

$$\|\mathcal{R}(f)\|_{WLM^{\alpha,V,\{x_0\}}_{1,\varphi_2}} \le C\|f\|_{LM^{\alpha,V,\{x_0\}}_{1,\varphi_1}}.$$

(ii) If $1 < p < p_0$, then the operator $\mathcal{R}$ is bounded from $LM^{\alpha,V,\{x_0\}}_{p,\varphi_1}$ to $LM^{\alpha,V,\{x_0\}}_{p,\varphi_2}$. Moreover, there exists a constant C such that

$$\|\mathcal{R}(f)\|_{LM^{\alpha,V,\{x_0\}}_{p,\varphi_2}} \le C\|f\|_{LM^{\alpha,V,\{x_0\}}_{p,\varphi_1}}.$$

(iii) If $p'_0 < p < \infty$, then the operator $\mathcal{R}^*$ is bounded from $LM^{\alpha,V,\{x_0\}}_{p,\varphi_1}$ to $LM^{\alpha,V,\{x_0\}}_{p,\varphi_2}$, where $p'_0 = \frac{p_0}{p_0-1}$. Moreover, there exists a constant C such that

$$\|\mathcal{R}^*(f)\|_{LM^{\alpha,V}_{p,\varphi_2}} \le C\|f\|_{LM^{\alpha,V}_{p,\varphi_1}}.$$

Corollary 1.1. *Let $V \in RH_q$ with $n/2 \le q < n$, $\alpha \ge 0$, $1/p_0 = 2/q_0 - 2/n$, q_0 is the reverse Hölder index of V, and $\varphi_1, \varphi_2 \in \Omega^{\alpha,V}_p$ satisfies the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x, s)s^{\frac{n}{p}}}{t^{\frac{n}{p}}} \frac{dt}{t} \le c_0\varphi_2(x, r), \tag{9}$$

where c_0 does not depend on x and r.

(i) If $p = 1$, then the operator $\mathcal{R}$ is bounded from $M^{\alpha,V}_{1,\varphi_1}$ to $WM^{\alpha,V}_{1,\varphi_2}$.

(ii) If $1 < p < p_0$, then the operator $\mathcal{R}$ is bounded from $M^{\alpha,V}_{p,\varphi_1}$ to $M^{\alpha,V}_{p,\varphi_2}$.

(iii) If $p'_0 < p < \infty$, then the operator $\mathcal{R}^*$ is bounded from $M^{\alpha,V}_{p,\varphi_1}$ to $M^{\alpha,V}_{p,\varphi_2}$, where $p'_0 = \frac{p_0}{p_0-1}$.

Theorem 1.2. *Let $x_0 \in \mathbb{R}^n$, $V \in RH_q$ with $n/2 \le q < n$, $\alpha \ge 0$, $1/p_0 = 2/q_0 - 2/n$, $b \in BMO_\theta(\rho)$, q_0 is the reverse Hölder index of V, and $\varphi_1, \varphi_2 \in \Omega^{\alpha,V}_{p,loc}$ satisfies the condition*

$$\int_r^\infty \left(1 + \ln\frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0, s)s^{\frac{n}{p}}}{t^{\frac{n}{p}}} \frac{dt}{t} \le c_0\varphi_2(x_0, r), \tag{10}$$

where c_0 *does not depend on* x *and* r.

(i) If $1 < p < p_0$, *then the commutator operator* $[b, \mathcal{R}]$ *is bounded from* $LM_{p,\varphi_1}^{\alpha,V,\{x_0\}}$ *to* $LM_{p,\varphi_2}^{\alpha,V,\{x_0\}}$. *Moreover, there exists a constant* C *such that*

$$\|[b, \mathcal{R}](f)\|_{LM_{p,\varphi_2}^{\alpha,V,\{x_0\}}} \leq C\|f\|_{LM_{p,\varphi_1}^{\alpha,V,\{x_0\}}}.$$

(ii) If $p_0' < p < \infty$, *then the commutator operator* $[b, \mathcal{R}^*]$ *is bounded from* $LM_{p,\varphi_1}^{\alpha,V,\{x_0\}}$ *to* $LM_{p,\varphi_2}^{\alpha,V,\{x_0\}}$ *and*

$$\|[b, \mathcal{R}^*](f)\|_{LM_{p,\varphi_2}^{\alpha,V,\{x_0\}}} \leq C[b]_\theta\|f\|_{LM_{p,\varphi_1}^{\alpha,V,\{x_0\}}},$$

where C *does not depend on* f.

Corollary 1.2. *Let* $V \in RH_q$ *with* $n/2 \leq q < n$, $\alpha \geq 0$, $1/p_0 = 2/q_0 - 2/n$, $b \in BMO_\theta(\rho)$, q_0 *is the reverse Hölder index of* V, *and* $\varphi_1, \varphi_2 \in \Omega_p^{\alpha,V}$ *satisfies the condition*

$$\int_r^\infty \Big(1 + \ln\frac{t}{r}\Big)\frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)s^{\frac{n}{p}}}{t^{\frac{n}{p}}}\frac{dt}{t} \leq c_0\varphi_2(x,r), \tag{11}$$

where c_0 *does not depend on* x *and* r.

(i) If $1 < p < p_0$, *then the operator* $[b, \mathcal{R}]$ *is bounded from* $M_{p,\varphi_1}^{\alpha,V}$ to $M_{p,\varphi_2}^{\alpha,V}$.

(ii) If $p_0' < p < \infty$, *then the operator* $[b, \mathcal{R}^*]$ *is bounded from* $M_{p,\varphi_1}^{\alpha,V}$ to $M_{p,\varphi_2}^{\alpha,V}$.

Theorem 1.3. *Let* $V \in RH_q$ *with* $n/2 \leq q < n$, $\alpha \geq 0$, $1/p_0 = 2/q_0 - 2/n$, q_0 *is the reverse Hölder index of* V, *and* $\varphi_1, \varphi_2 \in \Omega_{p,1}^{\alpha,V}$ *satisfies the conditions*

$$c_\delta := \int_\delta^\infty \sup_{x\in\mathbb{R}^n} \varphi_1(x,t)\frac{dt}{t} < \infty \tag{12}$$

for every $\delta > 0$, and

$$\int_r^\infty \varphi_1(x,t)\frac{dt}{t} \leq C_0\varphi_2(x,r), \tag{13}$$

where C_0 *does not depend on* $x \in \mathbb{R}^n$ *and* $r > 0$.

(i) If $1 < p < p_0$, *then the operator* $\mathcal{R}$ *is bounded from* $VM_{p,\varphi_1}^{\alpha,V}$ *to* $VM_{p,\varphi_2}^{\alpha,V}$.

(ii) If $p_0' < p < \infty$, *then the operator* $\mathcal{R}^*$ *is bounded from* $VM_{p,\varphi_1}^{\alpha,V}$ *to* $VM_{p,\varphi_2}^{\alpha,V}$.

Theorem 1.4. *Let* $V \in RH_q$ *with* $n/2 \leq q < n$, $\alpha \geq 0$, $1/p_0 = 2/q_0 - 2/n$, $b \in BMO_\theta(\rho)$, q_0 *is the reverse Hölder index of* V, *and* $\varphi_1, \varphi_2 \in \Omega_{p,1}^{\alpha,V}$ *satisfies the conditions*

$$\int_r^\infty \Big(1 + \ln\frac{t}{r}\Big)\varphi_1(x,t)\frac{dt}{t} \leq c_0\varphi_2(x,r), \tag{14}$$

where c_0 *does not depend on* x *and* r,

$$\lim_{r\to 0} \frac{\ln\frac{1}{r}}{\inf_{x\in\mathbb{R}^n} \varphi_2(x,r)} = 0 \tag{15}$$

and

$$c_\delta := \int_\delta^\infty \Big(1 + |\ln t|\Big) \sup_{x\in\mathbb{R}^n} \varphi_1(x,t)\frac{dt}{t} < \infty \tag{16}$$

for every $\delta > 0$.

(i) If $1 < p < p_0$, *then the operator* $[b, \mathcal{R}]$ *is bounded from* $VM_{p,\varphi_1}^{\alpha,V}$ *to* $VM_{p,\varphi_2}^{\alpha,V}$.

(ii) If $p_0' < p < \infty$, then the operator $[b,\mathcal{R}^*]$ is bounded from $VM^{\alpha,V}_{p,\varphi_1}$ to $VM^{\alpha,V}_{p,\varphi_2}$.

In this paper, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A \le CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2. Some preliminaries

We would like to recall the important properties concerning the critical function.

Lemma 2.1. *There exists constants $C > 0$ and $l_0 > 0$ such that*

$$\frac{1}{r^{n-2}} \int\limits_{B(x,r)} V(y)dy \le C\Big(1+\frac{r}{\rho(x)}\Big)^{l_0}.$$

Lemma 2.2. [26] *Let $V \in RH_{n/2}$. For the associated function ρ there exist C and $k_0 \ge 1$ such that*

$$C^{-1}\rho(x)\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{-k_0} \le \rho(y) \le C\rho(x)\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{\frac{k_0}{1+k_0}} \tag{17}$$

for all $x, y \in \mathbb{R}^n$.

Lemma 2.3. [2] *Suppose $x \in B(x_0, r)$. Then for $k \in N$ we have*

$$\frac{1}{\Big(1+\frac{2^k r}{\rho(x)}\Big)^{N}} \lesssim \frac{1}{\Big(1+\frac{2^k r}{\rho(x_0)}\Big)^{N/(k_0+1)}}.$$

We give some inequalities about the new BMO space $BMO_\theta(\rho)$.

Lemma 2.4. [4] *Let $1 \le s < \infty$. If $b \in BMO_\theta(\rho)$, then*

$$\Big(\frac{1}{|B|}\int\limits_B |b(y)-b_B|^s dy\Big)^{1/s} \le [b]_\theta \Big(1+\frac{r}{\rho(x)}\Big)^{\theta'}$$

for all $B = B(x,r)$, with $x \in \mathbb{R}^n$ and $r > 0$, where $\theta' = (k_0+1)\theta$ and k_0 is the constant appearing in (17).

Lemma 2.5. [4] *Let $1 \le s < \infty$, $b \in BMO_\theta(\rho)$, and $B = B(x,r)$. Then*

$$\Big(\frac{1}{|2^k B|}\int\limits_{2^k B} |b(y)-b_B|^s dy\Big)^{1/s} \le [b]_\theta k \Big(1+\frac{2^k r}{\rho(x)}\Big)^{\theta'}$$

for all $k \in \mathbb{N}$, with θ' as in Lemma 2.4.

Let K^* be the kernel of $\mathcal{R}^*$, then we have

Lemma 2.6. [4] *Let $V \in RH_q$, we have the following results*

(i) If $n/2 \le q < n$, then for every N, there exists a constant $C_N > 0$ such that

$$|K^*(x,z)| \le \frac{C_N\big(1+\frac{|x-z|}{\rho(x)}\big)^{-N}}{|x-z|^{n-2}} \left(\int\limits_{B(z,|x-z|/4)} \frac{V^2(u)}{|u-z|^{n-2}}du + \frac{1}{|x-z|^2}\right). \tag{18}$$

(ii) When $q \ge n$, the term involving V can be dropped from above formula.

The following results the estimates the L_p boundedness of the operators $\mathcal{R}$ and $\mathcal{R}^*$.

Lemma 2.7. [16] *Let $V \in RH_q$ with $n/2 \le q < n$, $1/p_0 = 2/q_0 - 2/n$.*

(i) If $1 < p < p_0$, then the operator $\mathcal{R}$ is bounded on $L_p(\mathbb{R}^n)$. Moreover, there exists a constant C such that

$$\|\mathcal{R}(f)\|_{L_p(\mathbb{R}^n)} \le C\|f\|_{L_p(\mathbb{R}^n)},$$

(ii) If $p = 1$, then the operator $\mathcal{R}$ is bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$. Moreover, there exists a constant C such that

$$\|\mathcal{R}(f)\|_{WL_1(\mathbb{R}^n)} \leq C\|f\|_{L_1(\mathbb{R}^n)}.$$

(iii) If $p_0' < p < \infty$, then the operator $\mathcal{R}^*$ is bounded on $L_p(\mathbb{R}^n)$. Moreover, there exists a constant C such that

$$\|\mathcal{R}^*(f)\|_{L_p(\mathbb{R}^n)} \leq C\|f\|_{L_p(\mathbb{R}^n)}.$$

The following results the estimates the L_p boundedness of the commutator operators $[b, \mathcal{R}]$ and $[b, \mathcal{R}^*]$.

Lemma 2.8. [17] *Let $V \in RH_q$ with $n/2 \leq q < n$, $1/p_0 = 2/q_0 - 2/n$ and $b \in BMO_\theta(\rho)$.*

(i) If $1 < p < p_0$, then the commutator operator $[b, \mathcal{R}]$ is bounded on $L_p(\mathbb{R}^n)$. Moreover, there exists a constant C such that

$$\|[b, \mathcal{R}](f)\|_{L_p(\mathbb{R}^n)} \leq C\|f\|_{L_p(\mathbb{R}^n)},$$

(ii) If $p_0' < p < \infty$, then the commutator operator $[b, \mathcal{R}^*]$ is bounded on $L_p(\mathbb{R}^n)$. Moreover, there exists a constant C such that

$$\|[b, \mathcal{R}^*](f)\|_{L_p(\mathbb{R}^n)} \leq C\|f\|_{L_p(\mathbb{R}^n)}.$$

We recall a relationship between essential supremum and essential infimum.

Lemma 2.9. [29] *Let f be a real-valued nonnegative function and measurable on E. Then*

$$\Big(\operatorname*{ess\,inf}_{x\in E} f(x)\Big)^{-1} = \operatorname*{ess\,sup}_{x\in E} \frac{1}{f(x)}.$$

3. Proof of Theorem 1.1.

To prove Theorem 1.1., we first investigate the following local estimate.

Theorem 3.1. *Let $V \in RH_q$ with $n/2 \leq q < n$ and $1/p_0 = 2/q_0 - 2/n$.*

(i) If $p = 1$, then the inequality

$$\|\mathcal{R}(f)\|_{WL_1(B(x_0,r))} \lesssim r^n \int_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^n} \frac{dt}{t} \tag{19}$$

holds for any $f \in L^{loc}(\mathbb{R}^n)$ and for any $x_0 \in \mathbb{R}^n$, $r > 0$.

(ii) If $1 < p < p_0$, then the inequality

$$\|\mathcal{R}(f)\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t} \tag{20}$$

holds for any $f \in L_p^{loc}(\mathbb{R}^n)$ and for any $x_0 \in \mathbb{R}^n$, $r > 0$.

(iii) If $p_0' < p < \infty$, then the inequality

$$\|\mathcal{R}^*(f)\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t} \tag{21}$$

holds for any $f \in L_p^{loc}(\mathbb{R}^n)$ and for any $x_0 \in \mathbb{R}^n$, $r > 0$.

Proof. Since the proofs for the case $1 < p < p_0$ and the case $p_0' < p < \infty$ are very similar, we only prove the case $p_0' < p < \infty$.

For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ and $\lambda B = B(x_0, \lambda r)$ for any $\lambda > 0$. We write f as $f = f_1 + f_2$, where $f_1(y) = f(y)\chi_{B(x_0,2r)}(y)$, and $\chi_{B(x_0,2r)}$ denotes the characteristic function of $B(x_0, 2r)$. Then

$$\|\mathcal{R}^*(f)\|_{L_p(B(x_0,r))} \leq \|\mathcal{R}^*(f_1)\|_{L_p(B(x_0,r))} + \|\mathcal{R}^*(f_2)\|_{L_p(B(x_0,r))}.$$

Since $f_1 \in L_p(\mathbb{R}^n)$ and from the boundedness of $\mathcal{R}^*$ on $L_p(\mathbb{R}^n)$, $p_0' < p < \infty$ it follows that

$$\|\mathcal{R}^*(f_1)\|_{L_p(B(x_0,r))} \lesssim \|f\|_{L_p(B(x_0,2r))} \lesssim r^{\frac{n}{p}} \|f\|_{L_p(B(x_0,2r))} \int_{2r}^{\infty} \frac{dt}{t^{\frac{n}{p}+1}}$$

$$\lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t}. \tag{22}$$

To estimate $\|\mathcal{R}^*(f_2)\|_{L_p(B(x_0,2r))}$ obverse that $x \in B$, $y \in (2B)^c$ implies $\frac{1}{2}|x_0 - y| \leq |x - y| \leq \frac{3}{2}|x_0 - y|$. Then by Lemma 2.7. for all $x \in B(x_0, r)$ we have

$$\begin{aligned}
|\mathcal{R}^*(f_2)(x)| &\leq \int_{(2B)^c} |K^*(x,y)f(y)|dy \lesssim \int_{(2B)^c} \frac{1}{(1+\frac{|x-y|}{\rho(x)})^N} \frac{|f(y)|}{|x-y|^n} dy \\
&+ \int_{(2B)^c} \frac{1}{(1+\frac{|x-y|}{\rho(x)})^N} \frac{|f(y)|}{|x-y|^{n-1}} dy \int_{B(y,|x-y|/4)} \frac{V(z)}{|z-y|^{n-1}} dz dy \\
&\lesssim \int_{(2B)^c} \frac{1}{(1+\frac{|x_0-y|}{\rho(x)})^N} \frac{|f(y)|}{|x_0-y|^n} dy \\
&+ \int_{(2B)^c} \frac{1}{(1+\frac{|x_0-y|}{\rho(x)})^N} \frac{|f(y)|}{|x_0-y|^{n-1}} dy \int_{B(y,|x_0-y|/4)} \frac{V(z)}{|z-y|^{n-1}} dz dy \\
&= I_1 + I_2.
\end{aligned}$$

By Hölder's inequality and Lemma 2.3. we get

$$\begin{aligned}
I_1 &\lesssim \frac{1}{(1+\frac{2r}{\rho(x)})^N} \int_{(2B)^c} \frac{|f(y)|}{|x_0-y|^n} dy \lesssim \frac{1}{(1+\frac{2r}{\rho(x)})^N} \sum_{k=1}^{\infty} (2^{k+1}r)^{-n} \int_{2^{k+1}B} |f(y)| dy \\
&\lesssim \frac{1}{(1+\frac{2r}{\rho(x_0)})^{N/(k_0+1)}} \sum_{k=1}^{\infty} \int_{2^k r}^{2^{k+1}r} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t} \\
&\lesssim \frac{1}{(1+\frac{2r}{\rho(x_0)})^{N/(k_0+1)}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t}.
\end{aligned} \tag{23}$$

For I_2, by Lemmas 2.1, 2.3. and Hölder's inequality we get

$$\begin{aligned}
I_2 &\lesssim \sum_{k=1}^{\infty} \frac{1}{(2^{k+1}r)^{n-1}} \frac{1}{(1+\frac{2^k r}{\rho(x)})^N} \int_{2^{k+1}B} |f(y)| dy \int_{B(x_0,2^{k+1}r)} \frac{V(z)}{|z-y|^{n-2}} dz \\
&\lesssim \sum_{k=1}^{\infty} \frac{1}{(2^{k+1}r)^{n-1}} \frac{1}{(1+\frac{2^k r}{\rho(x_0)})^{N/(k_0+1)}} \int_{2^{k+1}B} |f(y)| \mathcal{I}_2(V\chi_{B(x_0,2^{k+1}r)})(y) dy \\
&\lesssim \sum_{k=1}^{\infty} \frac{1}{(2^{k+1}r)^{n-1}} \frac{1}{(1+\frac{2^k r}{\rho(x_0)})^{N/(k_0+1)}} \|f\|_{L_p(B(x_0,2^{k+1}r))} \|\mathcal{I}_2(V\chi_{B(x_0,2^{k+1}r)})\|_{L_{p'}(\mathbb{R}^n)}.
\end{aligned}$$

Since $p_0' < p < \infty$, $1/p_0 = 2/s - 2/n$, we can select an appropriate number s such that $1/p' = 2/s - 2/n$. Note that $\mathcal{I}_2$ is bounded from $L_{s/2}(\mathbb{R}^n)$ to $L_{p'}(\mathbb{R}^n)$, and $V \in RH_s$, then by Lemma 2.1. we have

$$\begin{aligned}
\|\mathcal{I}_2(V\chi_{B(x_0,2^{k+1}r)})\|_{L_{p'}(\mathbb{R}^n)} &\lesssim \|V^2_{\chi_{B(x_0,2^{k+1}r)}}\|_{L_{s/2}(\mathbb{R}^n)} \\
&= |B(x_0,2^{k+1}r)|^{\frac{2}{s}} \left(\frac{1}{|B(x_0,2^{k+1}r)|} \int\limits_{B(x_0,2^{k+1})} V^s(z)dz \right)^{2/s} \\
&\lesssim |B(x_0,2^{k+1}r)|^{\frac{2}{s}-\frac{4}{n}} \left(\frac{1}{(2^{k+1}r)^{n-2}} \int\limits_{B(x_0,2^{k+1})} V(z)dz \right)^{2} \\
&\lesssim (2^{k+1}r)^{\frac{n}{p'}-1} \Big(1 + \frac{2^k r}{\rho(x_0)}\Big)^{2l_0}.
\end{aligned}$$

Thus we get

$$\begin{aligned}
I_2 &\lesssim \sum_{k=1}^{\infty} (2^{k+1}r)^{-\frac{n}{p}} \frac{1}{\big(1+\frac{2^k r}{\rho(x_0)}\big)^{(N/(k_0+1)-l_0)}} \|f\|_{L_p(B(x_0,2^{k+1}r))} \\
&\lesssim \frac{1}{\big(1+\frac{2r}{\rho(x_0)}\big)^{(N/(k_0+1)-2l_0)}} \sum_{k=1}^{\infty} (2^{k+1}r)^{-\frac{n}{p}} \|f\|_{L_p(B(x_0,2^{k+1}r))} \\
&\lesssim \frac{1}{\big(1+\frac{2r}{\rho(x_0)}\big)^{(N/(k_0+1)-2l_0)}} \int\limits_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t}.
\end{aligned}$$

So that $1/p' = 2/s - 2/n$ and $s < n$.

Combining the estimates for I_1 and I_2 we obtain

$$\sup_{x \in B(x_0,r)} |\mathcal{R}^*(f_2)(x)| \lesssim \frac{1}{\big(1+\frac{2r}{\rho(x_0)}\big)^{(N/(k_0+1)-2l_0)}} \int\limits_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t}. \tag{24}$$

Taking $N \geq 2l_0(k_0+1)$, then

$$\|\mathcal{R}^*(f_2)\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \int\limits_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t}$$

holds for $p_0 < p < \infty$.

Let $p = 1$. From the weak (1,1) boundedness of T and (4.6) it follows that:

This completes the proof of Theorem 3.1. □

Proof of Theorem 1.1. from Lemma 2.9., we have

$$\frac{1}{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s) s^{\frac{n}{p}}} = \operatorname*{ess\,sup}_{t<s<\infty} \frac{1}{\varphi_1(x_0,s) s^{\frac{n}{p}}}.$$

Note the fact that $\|f\|_{L_p(B(x_0,t))}$ is a nondecresing function of t, and $f \in M^{\alpha,V}_{p,\varphi_1}$, then

$$\frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}\|f\|_{L_p(B(x_0,t))}}{\operatorname*{ess\,inf}\limits_{t<s<\infty}\varphi_1(x_0,s)s^{\frac{n}{p}}} \lesssim \operatorname*{ess\,sup}\limits_{t<s<\infty}\frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}\|f\|_{L_p(B(x_0,t))}}{\varphi_1(x_0,s)s^{\frac{n}{p}}}$$

$$\lesssim \sup_{0<s<\infty}\frac{\left(1+\frac{s}{\rho(x_0)}\right)^{\alpha}\|f\|_{L_p(B(x_0,s))}}{\varphi_1(x_0,s)s^{\frac{n}{p}}} \lesssim \|f\|_{LM^{\alpha,V,\{x_0\}}_{p,\varphi_1}}. \tag{25}$$

Since $\alpha \geq 0$, and (φ_1,φ_2) satisfies the condition (9), then

$$\int_{2r}^{\infty}\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}}\frac{dt}{t} = \int_{2r}^{\infty}\frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}\|f\|_{L_p(B(x_0,t))}}{\operatorname*{ess\,inf}\limits_{t<s<\infty}\varphi_1(x_0,s)s^{\frac{n}{p}}}\frac{\operatorname*{ess\,inf}\limits_{t<s<\infty}\varphi_1(x_0,s)s^{\frac{n}{p}}}{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}t^{\frac{n}{p}}}\frac{dt}{t}$$

$$\lesssim \|f\|_{LM^{\alpha,V,\{x_0\}}_{p,\varphi_1}}\int_{r}^{\infty}\frac{\operatorname*{ess\,inf}\limits_{t<s<\infty}\varphi_1(x_0,s)s^{\frac{n}{p}}}{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha}t^{\frac{n}{p}}}\frac{dt}{t}$$

$$\lesssim \|f\|_{LM^{\alpha,V,\{x_0\}}_{p,\varphi_1}}\left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha}\int_{r}^{\infty}\frac{\operatorname*{ess\,inf}\limits_{t<s<\infty}\varphi_1(x_0,s)s^{\frac{n}{p}}}{t^{\frac{n}{p}}}\frac{dt}{t}$$

$$\lesssim \|f\|_{LM^{\alpha,V,\{x_0\}}_{p,\varphi_1}}\left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha}\varphi_2(x_0,r). \tag{26}$$

Then by Theorem 3.1. we get

$$\|\mathcal{R}^*(f)\|_{LM^{\alpha,V,\{x_0\}}_{p,\varphi_2}} \lesssim \sup_{r>0}\left(1+\frac{r}{\rho(x_0)}\right)^{\alpha}\varphi_2(x_0,r)^{-1}r^{-n/p}\|\mathcal{R}^*(f)\|_{L_p(B(x_0,r))}$$

$$\lesssim \sup_{r>0}\left(1+\frac{r}{\rho(x_0)}\right)^{\alpha}\varphi_2(x_0,r)^{-1}\int_{2r}^{\infty}\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}}\frac{dt}{t} \lesssim \|f\|_{LM^{\alpha,V,\{x_0\}}_{p,\varphi_1}}.$$

4. Proof of Theorem 1.2.

As the proof of Theorem 1.1., it suffices to prove the following result.

Theorem 4.1. *Let $V \in RH_q$ with $n/2 \leq q < n$, $\alpha \geq 0$, $1/p_0 = 2/q_0 - 2/n$ and $b \in BMO_\theta(\rho)$.*
(i) If $1 < p < p_0$, then the inequality

$$\|[b,\mathcal{R}(f)]\|_{L_p(B(x_0,r))} \lesssim [b]_\theta\, r^{\frac{n}{p}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}}\frac{dt}{t} \tag{27}$$

holds for any $f \in L^{loc}_p(\mathbb{R}^n)$ and for any $x_0 \in \mathbb{R}^n$, $r > 0$.
(iii) If $p_0' < p < \infty$, then the inequality

$$\|[b,\mathcal{R}^*(f)]\|_{L_p(B(x_0,r))} \lesssim [b]_\theta\, r^{\frac{n}{p}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}}\frac{dt}{t} \tag{28}$$

holds for any $f \in L^{loc}_p(\mathbb{R}^n)$ and for any $x_0 \in \mathbb{R}^n$, $r > 0$.

Proof. Since the proofs for the case $1 < p < p_0$ and the case $p_0' < p < \infty$ are very similar, we only prove the case $p_0' < p < \infty$.

We write f as $f = f_1 + f_2$, where $f_1(y) = f(y)\chi_{B(x_0,2r)(y)}$. Then

$$\|[b,\mathcal{R}^*](f)\|_{L_p(B(x_0,r))} \leq \|[b,\mathcal{R}^*](f_1)\|_{L_p(B(x_0,r))} + \|[b,\mathcal{R}^*](f_2)\|_{L_p(B(x_0,r))}.$$

By the boundedness of $[b,\mathcal{R}^*]$ on $L_p(\mathbb{R}^n)$, $p_0' < p < \infty$ and similar to the estimate of (22) we get

$$\|[b,\mathcal{R}^*](f_1)\|_{L_p(B(x_0,r))} \lesssim [b]_\theta \|f\|_{L_p(B(x_0,2r))}$$
$$\lesssim [b]_\theta r^{\frac{n}{p}} \int\limits_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t}. \qquad (29)$$

We now turn to deal with the term $\|[b,\mathcal{R}^*](f_2)\|_{L_p(B(x_0,r))}$. For any given $x \in B(x_0,r)$ we have

$$|[b,\mathcal{R}^*](f_2)(x)| \le |b(x)-b_{2B}|\,|\mathcal{R}^*(f_2)(x)| + |\mathcal{R}^*((b-b_{2B})f_2)(x)|.$$

By (24) we have

$$\sup_{x\in B(x_0,r)} |\mathcal{R}^*(f_2)(x)| \lesssim \frac{1}{\Big(1+\frac{2r}{\rho(x_0)}\Big)^{(N/(k_0+1)-l_0)}} \int\limits_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t}.$$

By Lemma 2.4.,

$$\|b-b_{2B}\|_{L_p(B(x_0,r))} \lesssim [b]_\theta \Big(1+\frac{2r}{\rho(x_0)}\Big)^{\theta}.$$

Then by Lemma 2.3., and taking $N \ge (k_0+1)\theta$ we get

$$\||b(x)-b_{2B}|\mathcal{R}^*(f_2)\|_{L_p(B(x_0,r))} \lesssim [b]_\theta r^{\frac{n}{p}} \Big(1+\frac{2r}{\rho(x_0)}\Big)^{\theta-N/(k_0+1)+l_0} \int\limits_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t}$$
$$\lesssim [b]_\theta r^{\frac{n}{p}} \int\limits_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t}. \qquad (30)$$

Finally, let us estimate $\|\mathcal{R}^*((b-b_{2B})f_2)\|_{L_p(B(x_0,r))}$. By (18), Lemma 2.2. and 2.3. we have

$$\sup_{x\in B(x_0,r)} |\mathcal{R}^*((b-b_{2B})f_2)(x)| \le \int\limits_{(2B)^c} |K^*(x,y)(b(y)-b_{2B})f(y)|dy$$
$$\lesssim \int\limits_{(2B)^c} \frac{|b(y)-b_{2B}|}{\Big(1+\frac{|x_0-y|}{\rho(x)}\Big)^N} \frac{f(y)}{|x_0-y|^n} dy + \int\limits_{(2B)^c} \frac{|b(y)-b_{2B}|}{\Big(1+\frac{|x_0-y|}{\rho(x)}\Big)^N} \frac{f(y)}{|x_0-y|^{n-2}}$$
$$\times \int\limits_{B(y,|x_0-y|/4)} \frac{V^2(z)}{|z-y|^{n-2}} dz dy = J_1 + J_2.$$

Note that

$$\int\limits_{2^{k+1}B} |b(y)-b_{2B}||f(y)|dy \lesssim \int\limits_{2^{k+1}B} |b(y)-b_{2^{k+1}B}||f(y)|dy + |b_{2^{k+1}B}-b_{2B}|$$
$$\times \int\limits_{2^{k+1}B} |f(y)|dy \lesssim [b]_\theta k \Big(1+\frac{2^k r}{\rho(x_0)}\Big)^{\theta'} (2^k r)^{\frac{n}{p'}} \|f\|_{L_p(B(x_0,2^{k+1}r))}.$$

Then, by Lemma 2.3. we get

$$J_1 \lesssim [b]_\theta \sum_{k=1}^{\infty} \frac{k}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{N/(k_0+1)-\theta'}} (2^k r)^{-\frac{n}{p}} \|f\|_{L_p(B(x_0,2^{k+1}r))}$$
$$\lesssim [b]_\theta \sum_{k=1}^{\infty} k (2^k r)^{-\frac{n}{p}} \|f\|_{L_p(B(x_0,2^{k+1}r))} \lesssim [b]_\theta \sum_{k=1}^{\infty} k \int_{2^k r}^{2^{k+1}r} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t}.$$

Since $2^k r \le t \le 2^{k+1} r$, then $k \approx \ln \frac{t}{r}$. Thus

$$J_1 \lesssim [b]_\theta \sum_{k=1}^{\infty} \int_{2^k r}^{2^{k+1}r} \ln \frac{t}{r} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t} \lesssim [b]_\theta \int_{2r}^{\infty} \left(1+\ln \frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t}.$$

Choosing $\tilde{p}$ and $\tilde{s}$ such that $p > \tilde{p}$, and $1/\tilde{p}' = 2/\tilde{s} - 2/n$, then

$$J_2 \lesssim \sum_{k=1}^{\infty} \frac{1}{(2^{k+1}r)^{n-1}} \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{N/(k_0+1)}}$$
$$\times \int_{2^{k+1}B} |b(y)-b_{2B}||f(y)| \mathcal{I}_2(V^2_{\chi_{B(x_0,2^{k+1})}})(y)dy$$
$$\lesssim \sum_{k=1}^{\infty} \frac{1}{(2^{k+1}r)^{n-1}} \frac{1}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{N/(k_0+1)}} \times \|(b-b_{2B})f\|_{L_{\tilde{p}}(B(x_0,2^{k+1}r))} \|\mathcal{I}_2(V^2_{\chi_{B(x_0,2^{k+1})}})\|_{L_{\tilde{p}'}(\mathbb{R}^n)}.$$

Since $\mathcal{I}_2$ is bounded from $L_{\tilde{s}/2}(\mathbb{R}^n)$ to $L_{\tilde{p}'}(\mathbb{R}^n)$, and $V \in RH_{\tilde{s}}$ we have

$$\|\mathcal{I}_2(V^2_{\chi_{B(x_0,2^{k+1})}})\|_{L_{\tilde{p}'}(\mathbb{R}^n)} \lesssim (2^{k+1}r)^{\frac{d}{\tilde{p}'}-1} \left(1+\frac{2^{k+1}r}{\rho(x_0)}\right)^{2l_0}.$$

Let $v = \frac{p\tilde{p}}{p-\tilde{p}}$, then

$$\|(b-b_{2B})f\|_{L_{\tilde{p}}(B(x_0,2^{k+1}r))} \lesssim \|f\|_{L_p(B(x_0,2^{k+1}r))} \|(b-b_{2B})f\|_{L_v(B(x_0,2^{k+1}r))}.$$

But

$$\|(b-b_{2B})\|_{L_v(B(x_0,2^{k+1}r))} \lesssim [b]_\theta k |2^{k+1}B|^{\frac{1}{\tilde{p}}-\frac{1}{p}} \left(1+\frac{2^k r}{\rho(x_0)}\right)^{\theta'}.$$

Then

$$J_2 \lesssim \sum_{k=1}^{\infty} \frac{[b]_\theta k}{\left(1+\frac{2^k r}{\rho(x_0)}\right)^{N/(k_0-1)-l_0-\theta'}} (2^{k+1}r)^{-\frac{n}{p}} \|f\|_{L_p(B(x_0,2^{k+1}r))}$$
$$\lesssim [b]_\theta \int_{2r}^{\infty} \left(1+\ln \frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t}.$$

Thus,

$$\|\mathcal{R}^*((b-b_{2B})f_2)\|_{L_2(B(x_0,r))} \lesssim [b]_\theta r^{\frac{n}{p}} \int_{2r}^{\infty} \left(1+\ln \frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t}. \tag{31}$$

Combining (29), (30) and (31), the proof of Theorem 4.1. is completed.

Proof of Theorem 1.2. *Since* $f \in LM^{\alpha,V,\{x_0\}}_{p,\varphi_1}$ *and* (φ_1, φ_2) *satisfies the condition* (10), *by* (26) *we have*

$$\begin{aligned}
&\int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}}\frac{dt}{t} \\
&= \int_{2r}^{\infty} \frac{\Big(1+\frac{t}{\rho(x_0)}\Big)^{\alpha}\|f\|_{L_p(B(x_0,t))}}{\operatorname*{ess\,inf}\limits_{t<s<\infty}\varphi_1(x_0,s)s^{\frac{n}{p}}}\Big(1+\ln\frac{t}{r}\Big)\frac{\operatorname*{ess\,inf}\limits_{t<s<\infty}\varphi_1(x_0,s)s^{\frac{n}{p}}}{\Big(1+\frac{t}{\rho(x_0)}\Big)^{\alpha}t^{\frac{n}{p}}}\frac{dt}{t} \\
&\lesssim \|f\|_{LM^{\alpha,V,\{x_0\}}_{p,\varphi_1}}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\frac{\operatorname*{ess\,inf}\limits_{t<s<\infty}\varphi_1(x_0,s)s^{\frac{n}{p}}}{\Big(1+\frac{t}{\rho(x_0)}\Big)^{\alpha}t^{\frac{n}{p}}}\frac{dt}{t} \\
&\lesssim \|f\|_{LM^{\alpha,V,\{x_0\}}_{p,\varphi_1}}\Big(1+\frac{r}{\rho(x_0)}\Big)^{-\alpha}\int_{r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\frac{\operatorname*{ess\,inf}\limits_{t<s<\infty}\varphi_1(x_0,s)s^{\frac{n}{p}}}{t^{\frac{n}{p}}}\frac{dt}{t} \\
&\lesssim \|f\|_{LM^{\alpha,V,\{x_0\}}_{p,\varphi_1}}\Big(1+\frac{r}{\rho(x_0)}\Big)^{-\alpha}\varphi_2(x_0,r). \qquad (32)
\end{aligned}$$

Then from Theorem 4.1. we get

$$\begin{aligned}
&\|[b,\mathcal{R}^*](f)\|_{LM^{\alpha,V,\{x_0\}}_{p,\varphi_2}} \\
&\lesssim \sup_{x_0\in\mathbb{R}^n, r>0}\Big(1+\frac{r}{\rho(x)}\Big)^{\alpha}\varphi_2(x_0,r)^{-1}r^{-n/p}\|[b,\mathcal{R}^*](f)\|_{L_p(B(x_0,r))} \\
&\lesssim [b]_\theta \sup_{x_0\in\mathbb{R}^n, r>0}\Big(1+\frac{r}{\rho(x_0)}\Big)^{\alpha}\varphi_2(x_0,r)^{-1}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}}\frac{dt}{t} \\
&\lesssim [b]_\theta\|f\|_{LM^{\alpha,V,\{x_0\}}_{p,\varphi_1}}.
\end{aligned}$$

□

5. Proof of Theorem 1.3.

The statement is derived from the estimate (21). The estimation of the norm of the operator, that is, the boundedness in the non-vanishing space, immediately follows from by Theorem 1.1. So we only have to prove that

$$\lim_{r\to0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}^{\alpha,V}_{p,\varphi_1}(f;x,r)=0,\ p_0'<p<\infty \Rightarrow \lim_{r\to0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}^{\alpha,V}_{p,\varphi_2}(\mathcal{R}^*(f);x,r)=0 \qquad (33)$$

and

$$\lim_{r\to0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}^{\alpha,V}_{p,\varphi_1}(f;x,r)=0,\ 1<p<p_0\ \Rightarrow \lim_{r\to0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}^{\alpha,V}_{p,\varphi_2}(\mathcal{R}(f);x,r)=0. \qquad (34)$$

To show that $\sup\limits_{x\in\mathbb{R}^n}\Big(1+\frac{r}{\rho(x)}\Big)^{\alpha}\varphi_2(x,r)^{-1}r^{-n/p}\|\mathcal{R}^*(f)\|_{L_p(B(x,r))}<\varepsilon$ for small r, we split the right-hand side of (21):

$$\Big(1+\frac{r}{\rho(x)}\Big)^{\alpha}\varphi_2(x,r)^{-1}r^{-n/p}\|\mathcal{R}^*(f)\|_{L_p(B(x,r))}\le C[I_{\delta_0}(x,r)+J_{\delta_0}(x,r)], \qquad (35)$$

where $\delta_0 > 0$ (we may take $\delta_0 > 1$), and

$$I_{\delta_0}(x,r) := \frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_2(x,r)} \int\limits_r^{\delta_0} t^{-\frac{n}{p}-1}\|f\|_{L_p(B(x,t))}dt$$

and

$$J_{\delta_0}(x,r) := \frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_2(x,r)} \int\limits_{\delta_0}^{\infty} t^{-\frac{n}{p}-1}\|f\|_{L_p(B(x,t))}dt$$

and it is supposed that $r < \delta_0$. We use the fact that $f \in VM^{\alpha,V}_{p,\varphi_1}(\mathbb{R}^n)$ and choose any fixed $\delta_0 > 0$ such that

$$\sup_{x\in\mathbb{R}^n} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \varphi_1(x,r)^{-1} r^{-n/p}\|f\|_{L_p(B(x,r))} < \frac{\varepsilon}{2CC_0},$$

where C and C_0 are constants from (13) and (35). This allows to estimate the first term uniformly in $r \in (0,\delta_0)$:

$$\sup_{x\in\mathbb{R}^n} CI_{\delta_0}(x,r) < \frac{\varepsilon}{2},\ 0 < r < \delta_0.$$

The estimation of the second term now my be made already by the choice of r sufficiently small. Indeed, thanks to the condition (6) we have

$$J_{\delta_0}(x,r) \le c_{\sigma_0}\ \frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_1(x,r)}\ \|f\|_{VM^{\alpha,V}_{p,\varphi_1}},$$

where c_{σ_0} is the constant from (2). Then, by (6) it suffices to choose r small enough such that

$$\sup_{x\in\mathbb{R}^n} \frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_2(x,r)} \le \frac{\varepsilon}{2c_{\sigma_0}\|f\|_{VM^{\alpha,V}_{p,\varphi_1}}},$$

which completes the proof of (33).

The proof of (34) is similar to the proof of (33).

6. Proof of Theorem 1.4.

The norm inequality having already been provided by Theorem Corollary 1.2., we only have to prove the implication

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \varphi_1(x,r)^{-1} r^{-n/p}\|f\|_{L_p(B(x,r))} = 0$$
$$\Longrightarrow \lim_{r\to 0}\sup_{x\in\mathbb{R}^n} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \varphi_2(x,r)^{-1} r^{-n/p}\|[b,\mathcal{R}^*(f)]\|_{L_p(B(x,r))} = 0. \tag{36}$$

To check that

$$\sup_{x\in\mathbb{R}^n} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \varphi_2(x,r)^{-1} r^{-n/p}\|[b,\mathcal{R}^*(f)]\|_{L_p(B(x,r))} < \varepsilon \quad \text{for small} \quad r,$$

we use the estimate (28):

$$\varphi_2(x,r)^{-1} r^{-n/p}\|[b,\mathcal{R}^*(f)]\|_{L_p(B(x,r))} \lesssim \frac{[b]_\theta}{\varphi_2(x,r)} \int_r^{\infty} \left(1+\ln\frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}}\frac{dt}{t}.$$

We take $r < \delta_0$ where δ_0 will be chosen small enough and split the integration:

$$\left(1+\frac{r}{\rho(x)}\right)^{\alpha} \varphi_2(x,r)^{-1} r^{-n/p}\|[b,\mathcal{R}^*(f)]\|_{L_p(B(x,r))} \le C[I_{\delta_0}(x,r) + J_{\delta_0}(x,r)], \tag{37}$$

where

$$I_{\delta_0}(x,r) := \frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_2(x,r)} \int_r^{\delta_0} \left(1+\ln\frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t}$$

and

$$J_{\delta_0}(x,r) := \frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_2(x,r)} \int_{\delta_0}^{\infty} \left(1+\ln\frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{p}}} \frac{dt}{t}.$$

We choose a fixed $\delta_0 > 0$ such that

$$\sup_{x\in\mathbb{R}^n} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \varphi_1(x,r)^{-1} r^{-n/p} \|f\|_{L_p(B(x,r))} < \frac{\varepsilon}{2CC_0}, \quad r \le \delta_0,$$

where C and C_0 are constants from (37) and (14), which yields the estimate of the first term uniform in $r \in (0,\delta_0) : \sup\limits_{x\in\mathbb{R}^n} C I_{\delta_0}(x,r) < \frac{\varepsilon}{2},\ 0 < r < \delta_0$.

For the second term, writing $1 + \ln\frac{t}{r} \le 1 + |\ln t| + \ln\frac{1}{r}$, we obtain

$$J_{\delta_0}(x,r) \le \frac{c_{\delta_0} + \widetilde{c_{\delta_0}}\ \ln\frac{1}{r}}{\varphi_2(x,r)} \|f\|_{M^{\alpha,V}_{p,\varphi_1}},$$

where c_{δ_0} is the constant from (16) with $\delta = \delta_0$ and $\widetilde{c_{\delta_0}}$ is a similar constant with omitted logarithmic factor in the integrand. Then, by (15) we can choose small r such that $\sup_{x\in\mathbb{R}^n} J_{\delta_0}(x,r) < \frac{\varepsilon}{2}$, which completes the proof.

7. Conclusion

In this paper, we obtain estimates for the higher order Riesz transforms $\mathcal{R}$, $\mathcal{R}^*$ and their commutators $[b,\mathcal{R}]$, $[b,\mathcal{R}^*]$ on local generalized Morrey space $LM^{\alpha,V,\{x_0\}}_{p,\varphi}(\mathbb{R}^n)$, generalized Morrey space $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ and vanishing generalized Morrey space $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ related to Schrödinger type operator.

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