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# ON COMPLETENESS OF A PART OF EIGEN AND ASSOCIATED VECTORS OF A QUADRATIC OPERATOR PENCIL FOR A DOUBLE-POINT BOUNDARY VALUE PROBLEM

S.S. MIRZOYEV[1,2], S.F. BABAYEVA[1,3]

Abstract. In the paper we study some spectral properties of a quadratic operator pencil, solvability of one type of double-point boundary value problem for elliptic type operator-differential equation. Here, at first analytic properties of the resolvent of a quadratic pencil, structure of the spectrum of the given operator pensil are studied. Then the completeness of a part of the system of eigen and associated vectors of the space of traces of regular solutions and also completeness of descending elementary solutions in the space of all regular solutions of a homogeneous equation, are proved. All obtained results are expressed in terms of the properties of the coefficients of the given quadratic pencil.

Keywords: operator pencil, completeness, the system of eigenvectors, regular solution, elementary solutions, resolvent.

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## 1. Introduction

Many problems of mechanics, mathematical physics, partial differential equations are reduced to studying the completeness of the system of eigen and associated vectors of quadratic pencils in different spaces [1, 4, 8, 10, 14, 17-21]. In [5, 6] first introduced the notion of multiple completeness of the system of eigen and associated vectors for higher order operator pencils and proved a theorem on multiple completeness of the system of eigen and associated vectors of one type of pencils called the Keldysh pencils. He connected the notion of multiple completeness of the system of eigen and associated vectors with the Cauchy problem in Hilbert space for corresponding operator-differential equation. Then, there appeared numerous works in this field (see e.g. [19] and references therein).

New promotion in the problems of completeness of eigen and associated vectors was obtained in [2-4]. In this paper, the multiple completeness of a part of eigen and associated vectors responding to eigen values from the left half-plane, was studied. He connected this problem with solvability of the initial boundary value problem in an infinite domain. Further, these results were developed in the works [9-16]. In [9-16] was suggested a new method for obtaining exact values of the norms of operators of intermediate derivatives in Sobolev-type spaces and used them for obtaining exact conditions of solvability of different boundary value problems in a semi-axis. After these works for operator-differential equations the completeness of the system of elementary solutions was obtained (see: e.g. [3, 12, 15]).

[1]Baku State University, Baku, Azerbaijan
[2]Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan
[3]Institute of Control Systems of NAS of Azerbaijan, Baku, Azerbaijan
e-mail: mirzoyevsabir@mail.ru, seva_babaeva@mail.ru
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In [10] the solvability conditions of one double-point boundary value problem were obtained and the obtained result was applied to solvability of one mixed problem for partial differential equations of elliptic type. In the paper we used the methods of the works [2, 3, 5, 6, 10, 12, 13, 15].

## 2. Some necessary notion and functional spaces

In a separable Hilbert space $H$ we consider a quadratic pencil of operators

$$P(\lambda) = -\lambda^2 E + \lambda A_1 + A_2 + A^2, \tag{1}$$

where $E$ is a unit operator in $H$, $\lambda$ is a spectral parameter, and the coefficients of the pencil (1) satisfy the conditions:

(1) $A$ is a positive-definite self-adjoint operator with completely continuous inverse $A^{-1}$;
(2) The operators $B_j = A_j\, A^{-j}$ $(j = 1,\, 2)$ are bounded in $H$.

If operator $A$ satisfies the condition (1), and $\{\ell_n\}_{n=1}^{\infty}$ is an orthonormed basis of eigen vectors, i.e. $A\,\ell_n = \lambda_n\,\ell_n$, $(\ell_n\,\ell_m) = \delta_{n,m} = \left\{\begin{array}{ll} 1, & n = m \\ 0, & n \neq m \end{array}\right.$, then the domain of definition of the operator $A^{\gamma}$ $(\gamma \geq 0)$

$$D(A^{\gamma}) = \left\{ x : \ x \in H; \ \sum_{n=1}^{\infty} \lambda_n^{2\gamma}\, |x,\, \ell_n|^2 < \infty \right\}$$

is a Hilbert space with respect to the scalar product $(x,\, y)_{\gamma} = (A^{\gamma}x,\, A^{\gamma}y)$, $x,\, y \in D(A^{\gamma})$. For $\gamma = 0$ we assume $H_0 = H$.

**Definition 2.1.** *If the equation $P(\mu_n)\, x_{0,n,j} = 0$ has a nonzero solution $x_{0,n,j} \in H_2$, then $\mu_n$ is called a characteristic numbers of the pencil $P(\lambda)$, while $x_{0,n,j}$ an eigen-vector responding to $\mu_n$. If the vectors $x_{0,n,j},\, x_{1,n,j},\, \ldots,\, x_{h,n,j} \in H_2$, $h = \overline{0,\, m_{n,j}}$, $j = \overline{1,\, q_n}$ satisfy the equations*

$$P(\mu_n)\, x_{h,n,j} + \ldots + P'(\mu_n)\, x_{h-1,n,j} + \frac{P''(\mu_n)}{2!}\, x_{h-2,n-j} = 0, h = \overline{0,\, m_{n,j}},$$

*then $x_{0,n,j},\, x_{1,n,j},\, \ldots,\, x_{h,n,j}$, $h = \overline{0,\, m_{n,j}}$, $j = \overline{1,\, q_n}$ are said to be eigen and associated vectors responding to $\mu_n$. If the system $\{x_{h,n,j}\}_{n=1}^{\infty}$, $h = \overline{0,\, m_{n,j}}$, $j = \overline{1,\, q_n}$ are eigen and associated vectors of the pencil, responding to $\mu_n$, then the functions*

$$u_{h,n,j}(t) = \ell^{\mu_n t} \left( \frac{t^h}{h!} x_{0,n,j} + \frac{t^{h-1}}{(h-1)!}\, x_{1,n,j} + \ldots + x_{h,n,j} \right), h = \overline{0,\, m_{n,j}}, q = \overline{1,\, j_n}$$

*satisfy the equation $P(d/dt)\, u(t) = 0$ and are called elementary solutions of the equation $P(d/dt)\, u(t) = 0$.*

We associate the operator pencil $P(\lambda)$ with the boundary value problem

$$P(d/dt)\, u(t) = -u''(t) + A^2\, u(t) + A_1\, u'(t) + A_2\, u(t) = 0, t \in R_+, R_+ = (0,\, \infty), \tag{2}$$

$$u(0) - \varepsilon\, u(1) = \varphi, \tag{3}$$

where $f(t)$, $u(t)$ are the functions determined in $R_+ = (0,\, \infty)$ almost everywhere with the values in $H$, $\varepsilon$ is generally speaking a complex number, $\varphi \in H$. Here and in the sequel, the derivatives are understood in the sense of distribution theory [1].

Denote by $L_2(R_+;\,H)$ Hilbert space of all functions $f(t)$, determined almost everywhere in $R_+ = (0,\,\infty)$, with the values in $H$, for which

$$\|f\|_{L_2(R_+;\,H)} = \left(\int_0^\infty \|f(t)\|^2\,dt\right)^{1/2} < \infty.$$

Following the monograph [7], we determine the Hilbert space

$$W_2^2(R_+;\,H) = \left\{u\,:\;\; u'' \in L_2(R_+;\,H)\,,\;\; A^2 u \in L_2(R_+;\,H)\right\}$$

with the norm

$$\|u\|_{W_2^2(R_+;\,H)} = \left(\left\|A^2 u\right\|^2_{L_2(R_+;\,H)} + \left\|u''\right\|^2_{L_2(R_+;\,H)}\right)^{1/2}.$$

By the theorem on intermediate derivatives it follows that if $u(t) \in W_2^2(R_+;\,H)$, then $A^{2-j}\,u^{(j)} \in L_2(R_+;\,H)$, $u^{(j)}(0) \in H_{2-j-1/2}$, $j = 0,\,1$, and

$$\left\|A^{2-j}\,u^{(j)}\right\|_{L_2(R_+;\,H)} \le const\;\|u\|_{W_2^2(R_+;\,H)}\,,$$

$$\left\|u^{(j)}(0)\right\|_{2-j-1/2} \le const\;\|u\|_{W_2^2(R_+;\,H)}\,, j = 0,\,1.$$

**Definition 2.2.** *If for $\varphi \in H_{3/2}$ there exists a function $u \in W_2^2(R_+;\,H)$ satisfying equation (2) almost everywhere in $R_+$, the boundary condition (2) in the sense of convergence*

$$\lim_{t\to+0}\|u(t) - \varepsilon\,u(1-t) - \varphi\|_{3/2} = 0$$

*and the estimation $\|u\|_{W_2^2(R_+;\,H)} \le const\;\|\varphi\|_{3/2}$ holds, then problem (2), (3) is said to be regularly solvable.*

In this paper we prove completeness of elementary decreasing solutions in the space of all regular solutions of problem (2), (3). To this end, at first we will study some spectral properties of the pencil $P(\lambda)$ and regular solvability of problem (2), (3).

Note that for $\varepsilon = 0$, problem (2), (3) was studied in the works [3, 12].

## 3. Some spectral properties of the operator pencil (1)

We have

**Lemma 3.1.** *Let conditions 1), 2) be fulfilled, and the operator $E + A_2\,A^{-2}$ be invertible in $H$. Then the operator pencil $P(\lambda)$ has only discrete spectrum with a unique limit point at infinity. If $A^{-1} \in \sigma_\rho$ $(0 < \rho < \infty)$, i.e. $\sum_{n=1}^{\infty}\lambda_n^{-\rho} < \infty$, then the operator-function $A^2\,P^{-1}(\lambda)$ is represented in the form of ratio of two entire functions of order not higher than $\rho$ and of minimal type with order $\rho$.*

*Proof.* As

$$P(\lambda) = \lambda^2 E + \lambda\,A_1 + A_2 + A^2 = \left(-\lambda^2 A^{-2} + E + \lambda\left(A_1 A^{-1}\right)\,A^{-1} + A_2 A^{-2}\right)A^2 =$$

$$= \left(-\lambda^2 A^{-2} + \lambda B_1\,A + E + B_2\right)\,A^2 =$$

$$= (E + B_2)\,\left(-\lambda^2\,(E + B_2)^{-1}\,A^{-2} + \lambda\,(E + B_2)^{-1}\,B_1 A^{-1} + E\right)\,A^2 =$$

$$= (E + B_2)\;L(\lambda)\;A^2,$$

where $L(\lambda) = -\lambda^2\,(E + B_2)^{-1}\,A^{-2} + \lambda\,(E + B_2)^{-1}\,B_1 A^{-1}$.

Obviously, coefficients of $L_2(\lambda)$ are a completely continuous operators for any $\lambda \in \mathbb{C}$, and $E + L(0) = E$ is invertible in $H$. Then by the Keldysh lemma [7] the operator pencil $E + L(\lambda)$ has only a discrete spectrum with a unique limit point at infinity. Then these properties refer to the pencil $P(\lambda)$ as well. On the other hand, if $A^{-1} \in \sigma_\rho$ $(0 < \rho < \infty)$, then the coefficients of $L(\lambda)$ with degrees $\lambda$ are the operators of $(E + B_2)^{-1} A^{-2} \in \sigma_{\rho/2}$, $(E + B_2)^{-1} B_1 A^{-1} \in \sigma_\rho$, therefore, by the Keldysh lemma [5, 6] $L(\lambda)$ is represented in the form of ratio of two entire functions of order not higher than $\rho$ and of minimal type with order $\rho$. As $P^{-1}(\lambda) = A^{-2} L^{-1}(\lambda)\ (E + B_2)^{-1}$, then the operator

$$A^2 P^{-1}(\lambda) = L^{-1}(\lambda)\ (E + B_2)^{-1}$$

the function also has such a property. The lemma is proved. □

Now let us prove a theorem on estimations of the resolvent on some pencils.

**Theorem 3.1.** *Let conditions 1), 2) be fulfilled, and let $\alpha \in (0, \pi/2]$. Then subject to the inequalities*

$$\frac{1}{2}\, \|B_1\| + \|B_2\| < \sin \alpha, \alpha \in (0, \pi/2]$$

*on the rays $\Gamma_{\pm\alpha} = \{\lambda :\ \arg \lambda = \pm\alpha\}$ there exists a resolvent $P^{-1}(\lambda)$ and on these rays we have the estimation*

$$|\lambda|^{2-\beta} \left\| A^\beta P^{-1}(\lambda) \right\| \le const, \beta \in [0, 2]. \tag{4}$$

*Proof.* Let $\lambda \in \Gamma_\alpha$, then $\lambda = r\, e^{i\,\alpha}$, $r > 0$ and

$$P(\lambda) = P\left(r\, e^{i\alpha}\right) = -r^2\, e^{2i\alpha} E + A^2 + r\, e^{i\alpha} A_1 + A_2 =$$

$$= \Big( E + r\, e^{i\alpha} A_1\, A^{-1} A \left(-r^2\, e^{2i\alpha} + A^2\right)^{-1} + A_2 A^{-2} A^2 \left(-r^2\, e^{2i\alpha} + \left(-r^2\, e^{i\alpha} E + A^2\right)\right)\Big) =$$

$$= (E + S) \cdot \left(-r^2\, e^{2i\alpha} + A^2\right). \tag{5}$$

On the other hand,

$$\|S\| = \left\| r\, e^{i\alpha} A_1\, A^{-1} A \left(-r^2\, e^{2i\alpha} + A^2\right)^{-1} + A_2 A^{-2} A^2 \left(-r^2\, e^{2i\alpha} + A^2\right)^{-1} \right\| \le$$

$$\le \|B_1\| \cdot \left\| r\, A \left(-r^2\, e^{2i\alpha} + A^2\right)^{-1} \right\| + \|B_2\| \cdot \left\| A^2 \left(-r^2\, e^{2\alpha} + A^2\right)^{-1} \right\|. \tag{6}$$

From the spectral expansion of the operator $A$ it follows that

$$\left\| r\, A \left(-r^2\, e^{2i\alpha} + A^2\right)^{-1} \right\| = \sup_{\lambda_n} \left| r\, \lambda_n\ \left(r^2 + e^{2i\alpha} + \lambda_n^2\right)^{-1} \right| =$$

$$= \sup_{\lambda_n} \left| r\, \lambda_n\ \left(r^4 + \lambda_n^4 - 2 r_n^2\, \lambda_n^2\, \cos 2\alpha\right)^{-1} \right| =$$

$$= \sup_{\lambda_n} \left| r\, \lambda_n\ \left(r^2 + \lambda_n^2\right)^2 - 2 r^2 \lambda_n^2\ (1 + \cos 2\alpha) \right| \le$$

$$\le \sup_{\lambda_n} \left| r\, \lambda^n\ \left(\left(r^2 + \lambda_n^2\right)^2 - \left(r^2 + \lambda_n^2\right)^2\, \cos^2 \alpha\right)^{1/2} \right| =$$

$$= \sup_{\lambda_n} \left| \lambda r^n\ \left(r_n^2 + \lambda_n^2\right) \right| \cdot \sin^{-1} \alpha \le \frac{1}{2}\, \sin^{-1} \alpha. \tag{7}$$

In the same way we have:

$$\left\| A_2\, A^{-2} A^2 \left(-r^2\, e^{2i\alpha} + A^2\right)^{-1} \right\| \le \|B_2\| \sup_{\lambda_n} \left| \lambda_n^2\ \left(\lambda_n^2 + \varepsilon^2\right)^{-1} \right| \cdot \sin^{-1} \alpha \le \|B_2\|\, \sin^{-1} \alpha. \tag{8}$$

Taking into account inequalities (7) and (8) in the inequality (6) we have

$$\|S\| \le \left(\frac{1}{2}\,\|B_1\| + \|B_2\|\right) \cdot \sin^{-1}\alpha < q < 1.$$

Then the operator $E + S$ is invertible, and from equality (5) we get

$$P^{-1}(\lambda) = \left(-r^2\,e^{2i\alpha} + A^2\right)^{-1}(E+S)^{-1},$$

therefore

$$|\lambda|^{2-\beta}\,\left\|A^{\beta}\,P^{-1}(\lambda)\right\| \le |\lambda|^{2-\beta}\,\left\|A^{\beta}\,\left(-r^2\,e^{2i\alpha} + A^2\right)^{-1}\right\| \times$$

$$\times\,\left\|(E+S)^{-1}\right\| \le \frac{1}{1-q}\,|\lambda|^{\beta}\,\left\|A^{\beta}\,\left(-r^2\,e^{2i\alpha} + A^2\right)^{-1}\right\|. \tag{9}$$

Obviously,

$$\left\|A^{\beta}\left(-r^2\,e^{2\alpha} + A^2\right)^{-1}\right\| = \sup_{\lambda_n}\left|\lambda_n^{\beta}\,\left(-r^2 e^{2i\alpha} + \lambda_n^2\right)^{-1}\right| =$$

$$= \sup_{\lambda_n}\left|\lambda_n^{\beta}\,\left(r^4 + \lambda_n^4 - 2r^2\lambda_n^2\right)^{-1}\right| \le$$

$$\le \sup_{\lambda_n}\left|\lambda_n^{\beta}\,\left(r^2 + \lambda_n^2\right)^{-1}\sin^{-1}\alpha\right| = \sup_{\lambda_n}\left|r\,\lambda^n\,\left(r_n^2 + \lambda_n^2\right)\right| \cdot \sin^{-1}\alpha \le \frac{1}{2}\,\sin^{-1}\alpha.$$

Consequently, from equality (9) it follows

$$|\lambda|^{2-\beta}\,\left|\lambda_n^{\beta}\,\left(-r^2 e^{2i\alpha} + \lambda_n^2\right)^{-1}\right| \le \left|r^{2-\beta}\lambda_n^{\beta}\,\left(r^2 + \lambda_n^2\right)^{-1}\sin^{-1}\alpha\right| \le$$

$$\le \sup_{\tau>0}\left(\tau^{2-\beta}\,\left(r^1 + 1\right)^{-1}\sin^{-1}\alpha\right) \le \begin{cases} 1,\ \beta = 0,\ \beta = 2, \\ \left(\frac{2-\beta}{2}\right)^{\frac{2-\beta}{2}} \cdot \left(\frac{\beta}{2}\right)^{\beta/2},\ \ \beta \in (0,\,2)\,. \end{cases}$$

Thus, from inequality (9) we have

$$|\lambda|^{2-\beta}\,\left\|A^{\beta}\,P^{-1}(\lambda)\right\| \le \frac{1}{1-q}\cdot d(\beta) = const,$$

where

$$d(\beta) = \begin{cases} 1, \quad \beta = 0,\ 2, \\ \left(\frac{2-\beta}{2}\right)^{\frac{2-\beta}{2}} \cdot \left(\frac{\beta}{2}\right)^{\beta/2},\ \ \beta \in (0,\,2)\,. \end{cases}$$

Theorem is proved □

From this theorem we have.

**Corollary 3.1.** *Let the conditions of theorem 3.1 be fulfilled. Then for sufficiently small* $\theta > 0$ *on the sectors* $S_{\pm\theta} = \left\{\lambda:\ \lambda = r\,e^{\pm i\,(\alpha+\delta)},\ |\delta| < \theta\right\}$ *there exists a resolvent* $P^{-1}(\lambda)$ *and estimations* (4) *hold on these sectors.*

*Proof.* For simplicity we prove the corollary for $\alpha = \pi/2$. Then

$$P(\lambda) = P(i\xi) + \xi^2\left(e^{2i\delta} - 1\right) + i\xi A_1\left(e^{i\delta} - 1\right) =$$

$$= P(i\xi) + \xi^2\left(e^{2i\delta} - 1\right) + i\xi B_1 A\left(e^{i\delta} - 1\right) =$$

$$= \left(E + \xi^2\left(e^{-2i\delta} - 1\right)\,P^{-1}(i\xi) + B_1 i\xi\left(e^{2i\delta} - 1\right)\right)\,P(i\xi)\,.$$

On the other hand, for sufficiently small $\theta$ and $|\delta| < \theta$

$$\left\|\xi^2\left(e^{2i\delta} - 1\right)\,P^{-1}(\lambda) + B_1 i\xi A\left(e^{i\delta} - 1\right)\right\| < \frac{1}{2}.$$

Then, it is obvious that $P(i\xi)$ exists on the sectors $S_{\pm\theta}$ and on these sectors

$$|\lambda|^{2-\beta}\left\|A^{\beta}P^{-1}(\lambda)\right\|\leq 2\,|\lambda|^{2-\beta}\left\|A^{\beta}P^{-1}(i\xi)\right\|\leq const.$$

The Corollary is proved. □

Let $K_o=\{x_{h,n,j}\}_{n=1}^{\infty}$, $h=\overline{0,\,m_{inj}}$, $j=\overline{1,\,q_n}$ be eigen and associated vectors corresponding to characteristic values $\mu_n$ from the left half-plane i.e. $R_e\mu_n<0$. Then, obviously, $x_{h,n,j}=u_{h,n,j}(0)\in H_{3/2}$. Let us construct the system

$$\psi_{h,n,j}=u_{h,n,j}(0)-\varepsilon\,u_{h,n,j}(1)\equiv x_{h,n,j}-\varepsilon\sum_{q=0}^{h,n,j}\left.\frac{\partial^q e^{\lambda}}{q!\partial\lambda^q}\right|_{\lambda=\lambda_i}x_{h-q,n,j}$$

and denote by $K_\varepsilon=\{\psi_{h,n,j}\}_{n=1}^{\infty}$, $h=\overline{1,\,m_{nj}}$, $j=\overline{1,\,q_n}$.

Our goal is to prove the completeness of the system $K_\varepsilon$ in the space $H_{3/2}$, and by means of this fact to obtain a theorem on elementary decreasing solutions of (1)-(3) in the space of obtained solutions of the problem (2)-(3).

## 4. On regular solvability of problem (2), (3)

Using the results of the paper [10], we will prove a theorem on regular solvability of problem (2), (3). At first we give the results that we need, from the paper [10].

**Lemma 4.1** (10)**.** *Let $\varphi\in H_{3/2}$. Then $e^{-tA}\varphi\in W_2^2(R_+;\,H)$, and*

$$\left\|A^2e^{-tA}\varphi\right\|_{L_2(R_+;\,H)}\leq\frac{1}{\sqrt{2}}\,\|\varphi\|_{3/2},$$

$$\left\|e^{-tA}\varphi\right\|_{W_2^2(R_+;\,H)}\leq\|\varphi\|_{3/2}.$$

**Theorem 4.1** (10)**.** *Let conditions 1), 2) be fulfilled, the operator $\left(E-\varepsilon\,e^{-A}\right)^{-1}$ have a bounded inverse determined in $H$, and*

$$q(\varepsilon)=C_1(\varepsilon)\,\|B_1\|+C_2(\varepsilon)\,\|B_2\|<1\quad\left(B_j=A_jA^{-j},\quad j=1,\,2\right),$$

*where*

$$C_1(\varepsilon)=\frac{1}{2}+\frac{|\varepsilon|}{\sqrt{2}}\left\|\left(E-e^{-A}\right)^{-1}\right\|=\frac{1}{2}+\frac{|\varepsilon|}{2}\sup_n\left|1-e^{-\lambda_n}\right|^{-1},$$

$$C_2(\varepsilon)=1+\frac{|\varepsilon|}{2}\left\|(E-e)^{-1}\right\|=1+\frac{|\varepsilon|}{2}\sup_n\left|1-e^{-\lambda_n}\right|^{-1}.$$

*Then the problem*

$$P(d/dt)\;w(t)=g(t)\,,t\in R_+,\tag{10}$$

$$w(0)\;-\varepsilon\,w(1)=0\tag{11}$$

*is regularly solvable.*

Regular solvability of problem (10), (11) means that for any $g(t)\in L_2(R_+;\,H)$ there exists a unique $w(t)\in W_2^2(R_+;\,H)$ that satisfies equation (10) almost everywhere in $R_+=(0,\,\infty)$ and boundary condition (11) in the sense of convergence

$$\lim_{t\to 0}\|w(t)-w(1-t)\|_{3/2}=0.$$

At first we consider the boundary value problem

$$P-u''(t)+A^2u(t)=0,t\in R_+,\tag{12}$$

$$u\,(0)\;-\varepsilon\,u\,(1)=\varphi. \tag{13}$$

We have

**Lemma 4.2.** *Let the operator* $E-\varepsilon\,e^{-A}$ *has a bounded inverse determined in the space* $H$*. Then the problem* (12), (13) *is regularly solvable.*

*Proof.* The general form of the solution (11) from the space $W_2^2\,(R_+;\,H)$ is of the form

$$u_o\,(t)=e^{-tA}x, x\in H_{3/2}.$$

From condition (12) it follows that $\left(E-\varepsilon\,e^{-A}\right)\,x=\varphi$, i.e. $x=\left(E-\varepsilon\,e^{-A}\right)^{-1}\,\varphi$. As

$$\|u\|_{W_2^2(R_+;\,H)}=\left\|e^{-tA}\,\left(E-e^{-A}\right)^{-1}\,\varphi\right\|_{W_2^2(R_+;\,H)}=\left\|\left(E-\varepsilon e^{-A}\right)^{-1}\right\|\cdot\left\|e^{-tA}\,\varphi\right\|_{W_2^2(R_+;\,H)}.$$

Applying lemma 4.1, we get $\|u\|_{W_2^2(R_+;\,H)}\le\left\|\left(E-\varepsilon\,e^{-A}\right)^{-1}\right\|\cdot\|\varphi\|_{3/2}$, i.e. problem (11), (12) is regularly solvable. □

We have

**Theorem 4.2.** *Let all conditions of Theorem 4.1 be fulfilled. Then, problem* (2), (3) *is regularly solvable.*

*Proof.* After substitution $u\,(t)=w\,(t)+e^{-tA}x$, where $w\,(t)\in W_2^2\,(R_+;\,H)$, while $u_o\,(t)=e^{-tA}x$ is a regular solution of problem (10), (11), with respect to $w\,(t)$ we get the following boundary value problem

$$P\,(d/dt)\;w\,(t)=g\,(t)\,,t\in R_+, \tag{14}$$

$$w\,(0)\;-\varepsilon\,w\,(1)=0, \tag{15}$$

as $x=\left(E-\varepsilon\,e^{-A}\right)^{-1}\varphi$, $x\in H_{3/2}$. Here the function $g\,(t)=A_1\,\frac{d}{dt}\,e^{-tA}x+A_2\,e^{-tA}$. Show that $g\,(t)\in L_2\,(R_+;\,H)$. As

$$\|g\,(t)\|_{L_2(R_+;\,H)}\le\left\|A_1\,A^{-1}\right\|\,\left\|A\,\frac{d}{dt}\,e^{-tA}x\right\|+\left\|A_2\,A^{-2}\right\|\,\left\|A^2\,e^{-tA}x\right\|_{L_2}=$$

$$=\left(\|B_1\|+\|B_2\|\right)\,\left\|A^2\,e^{-tA}x\right\|_{L_2(R_+;\,H)}.$$

Applying lemma 4.1, we get

$$\|g\,(t)\|_{L_2(R_+;H)}\le\frac{1}{\sqrt{2}}\,\left(\|B_1\|+\|B_2\|\right)\,\|x\|_{3/2}=$$

$$=\frac{1}{\sqrt{2}}\,\left(\|B_1\|+\|B_2\|\right)\,\left\|\left(E-\varepsilon\,e^{-A}\right)^{-1}\right\|\cdot\|\varphi\|_{3/2}<\infty,$$

i.e. $g\,(t)\in L_2\,(R_+;\,H)$. Then from theorem 4.1 it follows that problem (14), (15) is regularly solvable, and therefore, problem (2), (3) is regularly solvable.

The theorem is proved. □

## 5. On the completeness of the system $K_\varepsilon$

Using the results on the estimations of the norm of the resolvent on some rays and regular solvability, we will prove the completeness of the system $K_\varepsilon$in the space $H_{3/2}$ and completeness of the system of elementary solutions of regular decreasing solutions in the space of all regular solutions of problem (2), (3).

**Theorem 5.1.** *Let conditions 1), 2) be fulfilled, then the operator $E-\varepsilon\, e^{-A}$ has bounded inverse on the space $H$, and one of the following conditions holds:*

*a) $A^{-1}\in\sigma_\rho$ $(0<\rho<\infty)$ and we have the inequality*

$$C_1(\varepsilon)\ \|B_1\|+C_2(\varepsilon)\ \|B_2\|<\begin{cases}1, & 0<\rho\le 1,\\ \sin\frac{\pi}{2\rho}, & 1\le\rho<\infty,\end{cases}$$

*b) $A^{-1}\in\sigma_\rho$ $(0<\rho<\infty)$, the operators $B_1$ and $B_2$ are completely continuous in $H$, and*

$$C_1(\varepsilon)\ \|B_1\|+C_2(\varepsilon)\ \|B_2\|<1,$$

*where the numbers $C_1(\varepsilon)$ and $C_2(\varepsilon)$ are determined from theorem 4.1. Then the system $K_\varepsilon=\{\psi_{n,\,j,\,h}\}_{n=1}^{\infty}$, $j=\overline{1,\,q_n}$, $h=\overline{0,\,m_{nj}}$ is complete in the space $H_{3/2}$.*

*Proof.* At first prove that the system $\{x_{h,\,n,\,j}\}_{n=1}^{\infty}$, $h=\overline{0,\,m_{nj}}$, $j=\overline{1,\,q_n}$ is complete in $H_{3/2}$ (case $\varepsilon=0$). Obviously, if the system $\{x_{h,\,n,\,j}\}_{n=1}^{\infty}$, $h=\overline{0,\,m_{nj}}$, $j=\overline{1,\,q_n}$ is not complete in $H_{3/2}$, then there exists a vector $\varphi\in H_{3/2}$, such that $(x_{h,\,n,\,j},\,\psi)_{3/2}=0$, $n=\overline{1,\,\infty}$, $h=\overline{0,\,m_{nj}}$, $j=\overline{1,\,q_n}$. Then from expansion of the resolvent in the neighborhood of characteristic numbers it follows that the vector $\left(A^{3/2}\,P^{-1}(\lambda)\right)^*A^{3/2}$ will be holomorphic in the half-plane $\Pi=\{\lambda:\ Re\lambda<0\}$. If $u_0(t)$ is the solution of problem (12), (13) for $\varepsilon=0$, we can represent it in the form

$$u(t)=\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty}\hat{u}(\lambda)\ e^{\lambda t}d\lambda, \tag{16}$$

where $\hat{u}(\lambda)=P^{-1}(\lambda)\ ((\lambda\,E+A_1)\,u(0)+u'(0))$. From theorem 1 and its Corollary it follows that there exists a sufficiently small $\theta>0$ in the sectors $\Gamma_{\pm\theta}=\left\{\lambda:\lambda=r\,e^{\pm i\,(\pi/2+\delta)},0\le\delta<0,\right.$ $\left.r>0\right\}$, the resolvent has the estimation $\left\|P^{-1}(\lambda)\right\|\le const\ |\lambda|^{-2}$. Therefore, in formula (16) the integration contour may be replaced by the rays $\Gamma_{\pm\alpha}=\left\{\lambda:\ \lambda=r\,e^{\pm i\,(\pi/2+\theta)},\ r>0\right\}$. Then, for $t>0$

$$(u(t),\varphi)_{3/2}=\frac{1}{2\pi i}\int\limits_{\Gamma_{\pm\theta}}\Big((\lambda\,E+A_1)\,u(0)+u'(0)\,\Big(A^{3/2}\,P^{-1}(\lambda)\Big)^*A^{3/2}\varphi\Big)\ e^{\lambda t}d\lambda.$$

□

As $\hat{u}(\lambda)$ is the Laplacian transformation $u(t)\in W_2^2(R_+;\,H)$, then it is holomorphic in the right half-plane and has finite limit points on the imaginary axis. Therefore $\hat{u}(\lambda)$ is an entire function, and $\|\hat{u}(\lambda)\|\to 0$ as $\lambda\to\infty$ $(Re\lambda>0)$. In the case a), using the estimation $\|\hat{u}(\lambda)\|\le const\ |\lambda|^{1/2}$ (see theorem 1) and then applying the Phragmen-Lindelöf theorem, we get that $\hat{u}(\lambda)=a_0$, $a_0\in H$ then for $t>0$ $(u(t)\ ,\ \varphi)_{3/2}=0$. Passing to limit, we get that $\varphi=0$, i.e. the system $\{x_{h,\,n,\,j}\}_{n=1}^{\infty}$, $h=\overline{0,\,m_{nj}}$, $j=\overline{1,\,q_n}$ is complete in $H_{3/2}$. On the other hand, in the case b) this statement is obtained from the estimations of the resolvent in the angles whose angle is less than $\pi/\rho$. For proving the completeness of the system $\{\psi_{h,\,n,\,j}\}_{n=1}^{\infty}$, $j=\overline{1,\,q_n}$, $h=\overline{1,\,m_{nj}}$ we determine some bounded operator $T:\ H_{3/2}\ \to\ H_{3/2}$, that maps the system

$\{x_{h,\,n,\,j}\}_{n=1}^{\infty}$, $h=\overline{1,\,m_{nj}}$, $j=\overline{1,\,q_n}$ complete in $H_{3/2}$, onto the system $\{\psi_{h,\,n,\,j}\}_{n=1}^{\infty}$, $h=\overline{1,\,m_{nj}}$, $j=\overline{1,\,q_n}$.

Let $\varphi \in H_{3/2}$. Then subject to the conditions of the theorem, problem (2), (3) is regularly solvable for $\varepsilon=0$, i.e. for any $\varphi \in H_{3/2}$ there exists a regular solution of problem (2), (3) for $\varepsilon=0$ for which we have the inequality

$$C_1\,\|\varphi\|_{3/2} \le \|u_o\|_{W_2^2(R_+;\,H)} \le C_2\,\|\varphi\|_{3/2}\,.$$

Then, obviously, the vector $\psi = u\,(0)-\varepsilon\,u\,(1) \in H_{3/2}$ $(\varepsilon \neq 0)$. Then the equation $P\,(d/dt)\;u\,(t)=0$ with the initial condition $u\,(0)-\varepsilon\,u\,(1)=\psi$, has a regular solution $u_\varepsilon\,(t)$. Obviously,

$$d_1\,\|\psi\|_{3/2} \le \|u_\varepsilon\,(t)\| \le d_2\,\|\psi\|_{3/2}\,.$$

Now, determine the operator $T: H_{3/2} \to H_{3/2}$ in the following way: $T\,\varphi=\psi$. Obviously,

$$\|T\,\varphi\|_{3/2} = \|\psi\|_{3/2} \le const\,\|u_\varepsilon\,(t)\|_{W_2^2(R_+;\,H)} \le$$

$$\le const\,\|\psi\|_{3/2} \le const\,\Big(\|u\,(0)\|_{3/2} + \varepsilon\,\|u\,(1)\|_{3/2}\Big) \le$$

$$\le const\,\|u\|_{W_2^2(R_+;\,H)} \le const\,\|\varphi\|_{3/2}\,,$$

i.e. the operator $T$ is bounded. From the equation $T\,\varphi=0$ it follows that $\psi=0$, then $u_\varepsilon\,(t)=0$. Hence we get $\psi=0$. Then $u_\varepsilon\,(t)\equiv 0$, i.e. $u_\varepsilon\,(0)=0$. Then $\varphi=0$. Obviously $T:\ H_{3/2} \to H_{3/2}$ is an isomorphism, and

$$Tx_{h,\,n,\,j}=\psi_{h,\,n,\,j}, h=\overline{1,\,\infty}, j=\overline{1,\,q_n}, h=\overline{0,\,m_{nq}}$$

Hence we get that the system $\{\psi_{h,\,n,\,j}\}_{n=1}^{\infty}$ is complete in $H_{3/2}$.

From this theorem we get

**Theorem 5.2.** *Let the conditions of Theorem 3.1 be fulfilled. Then the system of elementary solutions of the homogeneous equation is complete in the space of regular solutions of problem* (2), (3).

*Proof.* As the system $\{\psi_{h,\,n,\,j}\}_{n=1}^{\infty}$, $h=\overline{0,\,m_n}$, $j=\overline{1,\,q_n}$ is complete in $H_{3/2}$, then for any $\varepsilon>0$ we can find a number $C_{\varepsilon,\,N}^{(P)}$ such that

$$\left\|\psi-\sum_{(P)}\sum_{h=\eta}^{N} C_{p,\,n}^{(h)}\,(\varepsilon)\;\psi_{h,\,n,\,j}\right\| < \varepsilon.$$

As $\psi=u\,(0)-\varepsilon\,u\,(1)$, and $\psi_{h,\,n,\,j}=u_{h,\,n,\,j}\,(0)-\varepsilon\,u_{h,\,n,\,j}\,(1)$, then

$$\left\|u\,(t)-\sum_{(p)}\sum_{n=1}^{N} C_{p,\,n}^{(h)}\,(\varepsilon)\;u_{n,\,h,\,j}(t)\right\|_{W_2^2(R_+;\,H)} \le \varepsilon\cdot const < \varepsilon_1,$$

where $\varepsilon_1>0$ is any number. □

The theorem is proved.

## 6. Conclusion

There is researched one type of double-point boundary value problem for elliptic type operator-differential equation. In the paper we study some spectral properties of a quadratic operator pencil, solvability of considered equation. At first we study analytic properties of the resolvent of a quadratic pencil, structure of the spectrum of the given operator pensil, the completeness of a part of the system of eigen and associated vectors of the space of traces of regular solutions. We also proved completeness of descending elementary solutions in the space of all regular solutions of a homogeneous equation. All obtained results are expressed by the properties of the coefficients of the given quadratic pencil.

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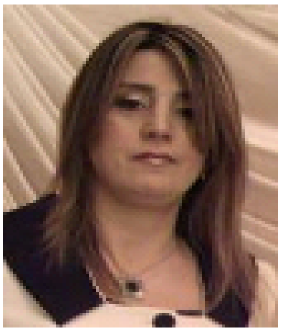
**Sabir S. Mirzoyev** is a Professor in department of Mathematical Analysis of the Baku State University. He got his Ph.D. degree from the Institute of Mathematics and Mechanics of ANAS in 1974, and his Doctor of Sciences degree from BSU in 1993. Main research interests: spectral theory, differential equations and their operator form.



**Sevinj F. Babayeva** graduated from Faculty of Mechanics-Mathematics of the Baku State University in 1990. She received her Ph.D. degree in Mathematics from the Institute of Mathematics and Mechanics of ANAS. Presently she is an assistant professor of the Department "Theory of Function and Functional Analysis" at the Baku State University.