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PARABOLIC NON-SINGULAR INTEGRAL OPERATOR AND ITS COMMUTATORS ON PARABOLIC VANISHING GENERALIZED ORLICZ-MORREY SPACES

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Abstract. We obtain the sufficient conditions for the boundedness of the parabolic non-singular integral operator and its commutators on the parabolic vanishing generalized Orlicz-Morrey spaces $M^{\Phi,\varphi}(\mathbb{D}_+^{n+1})$ including their weak versions.

Keywords: parabolic vanishing generalized Orlicz-Morrey spaces, parabolic non-singular integral, commutator.

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1. Introduction

In connection with elliptic partial differential equations, C. Morrey proposed a weak condition for the solution to be continuous enough in [37]. Later on, his condition became a family of the normed spaces which are called Morrey spaces. Although the notion is originally from the partial differential equations, the space turned out to be important in many branches of mathematics. Although such spaces allow to describe local properties of functions better than Lebesgue spaces, they have some unpleasant issues. It is well-known that Morrey spaces are non-separable and that the usual classes of nice functions are not dense in such spaces. Moreover, various Morrey spaces are defined in the process of study. Guliyev et al. [16, 36, 38] introduced and study the boundedness of some classical integral operators in the generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$.

It is well-known that the commutator is an important integral operator and it plays a key role in harmonic analysis. In 1965, Calderón [4, 5] studied a kind of commutators, appearing in Cauchy integral problems of Lip-line. Let T be a Calderón-Zygmund singular integral operator and $b \in BMO(\mathbb{R}^n)$. A celebrated result of Coifman et al. [8] states that the commutator operator $[b,T]f = T(bf) - bTf$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. The commutator of Calderón-Zygmund operators plays an important role in the study of regularity of solutions of elliptic partial differential equations of second order (see, for example, [6, 7, 10, 20, 21, 34]).

In [9], the generalized Orlicz-Morrey space $M^{\Phi,\varphi}(\mathbb{R}^n)$ was introduced to unify Orlicz and generalized Morrey spaces. Other definitions of generalized Orlicz-Morrey spaces can be found in [39] and [48]. In words of [24], the generalized Orlicz-Morrey space is the third kind and the ones in [39] and [48] are the first kind and the second kind, respectively. According to the examples in [15], one can say that the generalized Orlicz-Morrey spaces of the first kind and the second kind are different and that second kind and third kind are different. However, we do not know the relation between the first and the second kind.

Note that, Orlicz-Morrey spaces unify Orlicz and generalized Morrey spaces. We extend some results on generalized Morrey space in the papers [1, 11, 17, 18, 19, 25, 28] to the case of Orlicz-Morrey space in [9, 22, 23, 24].

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As based on the results of [17, 18], the following conditions were introduced in [9] (see, also [22]) for the boundedness of the singular integral operators on $M^{\Phi,\varphi}(\mathbb{R}^n)$,

$$\int_r^\infty \left(\operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-n}\big)} \right) \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t} \le C\, \varphi_2(x,r), \tag{1}$$

where C does not depend on x and r.

Consider the half-space $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$. For $x=(x',t) \in \mathbb{R}^{n+1}_+$, $x=(x'',x_n,t) \in \mathbb{D}^{n+1}_+ = \mathbb{R}^{n-1} \times \mathbb{R}_+ \times \mathbb{R}_+$, $\mathbb{D}^{n+1}_- = \mathbb{R}^{n-1} \times \mathbb{R}_- \times \mathbb{R}_+$. In the following, besides the standard parabolic metric $\varrho(x) = \max(|x'|, |t|^{1/2})$ we use the equivalent one $\rho(x) = \left(\frac{|x'|^2+\sqrt{|x'|^4+4t^2}}{2}\right)^{1/2}$ introduced by Fabes and Riviére in [14]. The induced by it topology consists of ellipsoids (parabolic balls)

$$\mathcal{E}_r(x) = \left\{ y \in \mathbb{R}^{n+1} : \ \frac{|x'-y'|^2}{r^2} + \frac{|t-\tau|^2}{r^4} < 1 \right\},\ |\mathcal{E}_r| = Cr^{n+2}.$$

It is easy to see that $\mathcal{E}_1(x)$ and $\mathbb{S}^n$ are the unit ball and the unit sphere, respectively, with respect to the both metrics and $\rho(x)$. On the other hand, the equivalence between the both parabolic metrics $\varrho(x)$ and $\rho(x)$ follows by the inclusion: for each $\mathcal{E}_r$ there exist parabolic cylinders $\underline{\mathcal{C}}$ and $\overline{\mathcal{C}}$ with measure comparable with r^{n+2} such that $\underline{\mathcal{C}} \subset \mathcal{E}_r \subset \overline{\mathcal{C}}$. In what follows all estimate obtained over ellipsoids hold true also over parabolic cylinders and we shall use this property without explicit references.

Let $\widetilde{x} = (x'', -x_n, t)$ be the "reflected point". The parabolic non-singular integral operator $\mathcal{R}$ is defined by (see [3])

$$\mathcal{R}f(x) = \int_{\mathbb{D}^{n+1}_+} \frac{|f(y)|}{\rho(\widetilde{x}-y)^{n+2}}\, dy. \tag{2}$$

The commutators generated by $b \in L^1_{\rm loc}(\mathbb{D}^{n+1}_+)$ and the operator $\mathcal{R}$ are defined by

$$[b,\mathcal{R}]f(x) = \int_{\mathbb{D}^{n+1}_+} \frac{b(x)-b(y)}{\rho(\widetilde{x}-y)^{n+2}}\, f(y)\, dy.$$

The operator $\mathcal{R}$ and its commutator appear in [3] in connection with boundary estimates for solutions to parabolic equations.

In [42, 43] we have studied the boundedness of the parabolic non-singular integral operator $\mathcal{R}$ on Orlicz and generalized Orlicz-Morrey spaces, respectively. Quite recently, we have also studied in [44] the boundedness of the commutator of parabolic non-singular integral operator $[b,\mathcal{R}]$ on parabolic generalized Orlicz-Morrey spaces of the third kind $M^{\Phi,\varphi}(\mathbb{D}^{n+1}_+)$ with BMO functions (see also [12]).

The main purpose of this paper is to find sufficient conditions on general Young function Φ and functions φ_1, φ_2 which ensure the boundedness of parabolic non-singular integral operator $\mathcal{R}$ from one parabolic vanishing generalized Orlicz-Morrey spaces $VM^{\Phi,\varphi_1}(\mathbb{D}^{n+1}_+)$ (definition see section 2) to another $VM^{\Phi,\varphi_2}(\mathbb{D}^{n+1}_+)$, from $VM^{\Phi,\varphi_1}(\mathbb{D}^{n+1}_+)$ to parabolic vanishing weak generalized Orlicz-Morrey spaces $VWM^{\Phi,\varphi_2}(\mathbb{D}^{n+1}_+)$ and the boundedness of commutator of the parabolic non-singular integral operator $[b,\mathcal{R}]$ from $VM^{\Phi,\varphi_1}(\mathbb{D}^{n+1}_+)$ to $VM^{\Phi,\varphi_2}(\mathbb{D}^{n+1}_+)$.

The following results are the fundamental theorems in this paper:

Theorem 1.1. *Let Φ be a Young function with $\Phi \in \Delta_2$. Let also $\varphi_1, \varphi_2 \in \Omega_{\Phi,1}$ satisfy*

$$c_\delta := \int_\delta^\infty \sup_{x \in \mathbb{D}^{n+1}_+} \varphi_1(x,t) \frac{dt}{t} < \infty, \tag{3}$$

for every $\delta > 0$*, and*

$$\frac{1}{\varphi_2(x,r)} \int_r^\infty \varphi_1(x,t)\,\frac{dt}{t} \le C_0, \tag{4}$$

where C_0 *does not depend on* $x \in \mathbb{D}_+^{n+1}$ *and* $r > 0$*. Then the parabolic non-singular integral operator* $\mathcal{R}$ *is bounded from* $VM^{\Phi,\varphi_1}(\mathbb{R}_+^{n+1})$ *to* $VWM^{\Phi,\varphi_2}(\mathbb{R}_+^{n+1})$*. If, in addition,* $\Phi \in \nabla_2$*, then the operator* $\mathcal{R}$ *is bounded from* $VM^{\Phi,\varphi_1}(\mathbb{D}_+^{n+1})$ *to* $VM^{\Phi,\varphi_2}(\mathbb{D}_+^{n+1})$*.*

Theorem 1.2. *Let* Φ *be a Young function with* $\Phi \in \Delta_2 \cap \nabla_2$*,* $b \in BMO(\mathbb{D}_+^{n+1})$*.* $\varphi_1, \varphi_2 \in \Omega_{\Phi,1}$ *satisfy*

$$\int_r^\infty \Big(1 + \ln\frac{t}{r}\Big)\varphi_1(x,t)\,\frac{dt}{t} \le C_0\varphi_2(x,r), \tag{5}$$

where C_0 *does not depend on* $x\mathbb{D}_+^{n+1}$ *and* $r > 0$*, and the conditions*

$$\lim_{r\to 0} \frac{\ln\frac{1}{r}}{\inf_{x\in\mathbb{D}_+^{n+1}} \varphi_2(x,r)} = 0, \tag{6}$$

and

$$c_\delta := \int_\delta^\infty (1 + |\ln t|) \sup_{x\in\mathbb{D}_+^{n+1}} \varphi_1(x,t)\frac{dt}{t} < \infty, \tag{7}$$

for every $\delta > 0$*. Then the commutator of the parabolic non-singular integral operator* $[b,\mathcal{R}]$ *is bounded from* $VM^{\Phi,\varphi_1}(\mathbb{D}_+^{n+1})$ *to* $VM^{\Phi,\varphi_2}(\mathbb{D}_+^{n+1})$*.*

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Definitions and preliminary results

2.1. **On Young Functions and Orlicz Spaces.** We recall the definition of Young functions.

Definition 2.1. A function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r\to\infty} \Phi(r) = \infty$.

From the convexity and $\Phi(0) = 0$, it follows that any Young function is increasing. If there exists $s \in (0,\infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for $r \ge s$. The set of Young functions such that

$$0 < \Phi(r) < \infty, \qquad \text{for} \qquad 0 < r < \infty,$$

will be denoted by $\mathcal{Y}$. If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0,\infty)$ and bijective from $[0,\infty)$ to itself.

For a Young function Φ and $0 \le s \le \infty$, let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\}.$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. We note that

$$\Phi(\Phi^{-1}(r)) \le r \le \Phi^{-1}(\Phi(r)) \quad \text{ for } 0 \le r < \infty.$$

It is well-known that

$$r \le \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \le 2r \qquad \text{for } r \ge 0, \tag{8}$$

where $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty), \} & , \quad r \in [0,\infty) \\ \infty & , \quad r = \infty. \end{cases}$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted also as $\Phi \in \Delta_2$, if

$$\Phi(2r) \le k\Phi(r) \text{for } r > 0,$$

for some $k > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2k}\Phi(kr), \qquad r \ge 0,$$

for some $k > 1$.

Definition 2.2. (Orlicz Space). For a Young function Φ, the set

$$L^{\Phi}(\mathbb{R}^{n+1}_{+}) = \left\{ f \in L^1_{\rm loc}(\mathbb{R}^{n+1}_{+}) : \int\limits_{\mathbb{R}^{n+1}_{+}} \Phi(k|f(x)|)dx < \infty \text{ for some } k > 0 \ , \right\}$$

is called Orlicz space. If $\Phi(r) = r^p$, $1 \le p < \infty$, then $L^{\Phi}(\mathbb{R}^{n+1}_{+}) = L^p(\mathbb{R}^{n+1}_{+})$. If $\Phi(r) = 0$, $(0 \le r \le 1)$ and $\Phi(r) = \infty$, $(r > 1)$, then $L^{\Phi}(\mathbb{R}^{n+1}_{+}) = L^{\infty}(\mathbb{R}^{n+1}_{+})$. The space $L^{\Phi}_{\rm loc}(\mathbb{R}^{n+1}_{+})$ is defined as the set of all functions f such that $f\chi_{\mathcal{E}} \in L^{\Phi}(\mathbb{R}^{n+1}_{+})$ for all parabolic balls $\mathcal{E} \subset \mathbb{R}^{n+1}_{+}$.

$L^{\Phi}(\mathbb{R}^{n+1}_{+})$ is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi}(\mathbb{R}^{n+1}_{+})} = \inf \left\{ \lambda > 0 : \int\limits_{\mathbb{R}^{n+1}_{+}} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)dx \le 1 \right\}.$$

We note that

$$\int\limits_{\mathbb{R}^{n+1}_{+}} \Phi\Big(\frac{|f(x)|}{\|f\|_{L^{\Phi}(\mathbb{R}^{n+1}_{+})}}\Big)dx \le 1. \tag{9}$$

The weak Orlicz space

$$WL^{\Phi}(\mathbb{R}^{n+1}_{+}) = \{f \in L^1_{\rm loc}(\mathbb{R}^{n+1}_{+}) : \|f\|_{WL^{\Phi}(\mathbb{R}^{n+1}_{+})} < +\infty\},$$

is defined by the norm

$$\|f\|_{WL^{\Phi}(\mathbb{R}^{n+1}_{+})} = \inf \Big\{ \lambda > 0 \ : \ \sup_{t>0} \Phi(t)\, m\Big(\frac{f}{\lambda},\ t\Big) \ \le 1 \Big\}.$$

2.2. **Parabolic vanishing generalized Orlicz-Morrey Space.** Various versions of generalized Orlicz-Morrey spaces were introduced in [39], [48] and [9]. We used the definition of [9] which runs as follows.

We now define the parabolic generalized Orlicz-Morrey spaces of the third kind. The *parabolic generalized Orlicz-Morrey space $M^{\Phi,\varphi}(\mathbb{R}^{n+1}_{+})$ of the third kind* is defined as the set of all measurable functions f for which the norm

$$\|f\|_{M^{\Phi,\varphi}(\mathbb{R}^{n+1}_{+})} \equiv \sup_{x \in \mathbb{R}^{n+1}_{+},\, r>0} \frac{1}{\varphi(x,r)} \Phi^{-1}\left(\frac{1}{|\mathcal{E}^{+}(x,r)|}\right) \|f\|_{L^{\Phi}(\mathcal{E}^{+}(x,r))},$$

is finite, where $\mathcal{E}^{+}(x,r) = B(x,r) \cap \mathbb{R}^{n+1}_{+}$. Also by $WM^{\Phi,\varphi}(\mathbb{R}^{n+1}_{+})$ we denote the *weak parabolic generalized Orlicz-Morrey space of the third kind* of all functions $f \in WL^{\Phi}_{\rm loc}(\mathbb{R}^{n+1}_{+})$ for which

$$\|f\|_{WM^{\Phi,\varphi}(\mathbb{R}^{n+1}_{+})} = \sup_{x \in \mathbb{R}^{n+1}_{+}, r>0} \varphi(x,r)^{-1}\Phi^{-1}(|\mathcal{E}^{+}(x,r)|^{-1})\, \|f\|_{WL^{\Phi}(\mathcal{E}^{+}(x,r))} < \infty,$$

where $WL^{\Phi}(\mathcal{E}^{+}(x,r))$ denotes the weak L^{Φ}-space of measurable functions f for which

$$\|f\|_{WL^{\Phi}(\mathcal{E}^{+}(x,r))} \equiv \|f\chi_{\mathcal{E}^{+}(x,r)}\|_{WL^{\Phi}(\mathbb{R}^{n+1}_{+})}.$$

Note that $M^{\Phi,\varphi}(\mathbb{R}^{n+1}_+)$ covers many classical function spaces.

Example 2.1. *Let $1 \le q \le p < \infty$ and $\Phi \in \Delta_2 \cap \nabla_2$. From the following special cases, we see that our results will cover the Lebesgue space $L^p(\mathbb{R}^{n+1}_+)$, the classical parabolic Morrey space $M^p_q(\mathbb{R}^{n+1}_+)$, the parabolic generalized Morrey space $M^{p,\varphi}(\mathbb{R}^{n+1}_+)$ and the Orlicz space $L^\Phi(\mathbb{R}^{n+1}_+)$ with norm coincidence:*

1. *If $\Phi(t) = t^p$ and $\varphi(t) = t^{-\frac{n}{p}}$, then $M^{\Phi,\varphi}(\mathbb{R}^{n+1}_+) = L^p(\mathbb{R}^{n+1}_+)$ with norm equivalence.*
2. *If $\Phi(t) = t^q$ and $\varphi(t) = t^{-\frac{n}{p}}$, then $M^{\Phi,\varphi}(\mathbb{R}^{n+1}_+)$, which is denoted by $M^p_q(\mathbb{R}^{n+1}_+)$, is the classical parabolic Morrey space.*
3. *If $\Phi(t) = t^p$, then $M^{\Phi,\varphi}(\mathbb{R}^{n+1}_+) = M^{p,\varphi}(\mathbb{R}^{n+1}_+)$ is the parabolic generalized Morrey space which were discussed in* [17, 36, 38].
4. *If $\varphi(t) = \Phi^{-1}(t^{-n})$, then $M^{\Phi,\varphi}(\mathbb{R}^{n+1}_+) = L^\Phi(\mathbb{R}^{n+1}_+)$, which is beyond the reach of parabolic generalized Orlicz-Morrey spaces of the second kind defined in* [15] *according to an example constructed in* [48].

Other definitions of generalized Orlicz-Morrey spaces can be found in [15, 39, 40, 41]; Therefore, our definition of generalized Orlicz-Morrey spaces here is named "third kind".

In the case $\varphi(x,r) = \frac{\Phi^{-1}\left(|\mathcal{E}^+(x,r)|^{-1}\right)}{\Phi^{-1}\left(|\mathcal{E}^+(x,r)|^{-\lambda/n}\right)}$, we get the parabolic Orlicz-Morrey space $\mathcal{M}^{\Phi,\lambda}(\mathbb{R}^{n+1}_+)$ from the parabolic generalized Orlicz-Morrey space $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^{n+1}_+)$. We refer to [22, Lemmas 2.8 and 2.9] for more information on the Orlicz-Morrey spaces.

Lemma 2.1. [22, Lemma 2.12] *Let Φ be a Young function and φ be a positive measurable function on $\mathbb{R}^{n+1}_+ \times (0,\infty)$.*

(i) *If*

$$\sup_{t<r<\infty} \frac{\Phi^{-1}(|\mathcal{E}^+(x,r)|^{-1})}{\varphi(x,r)} = \infty \quad \text{for some } t>0 \text{ and for all } x \in \mathbb{R}^{n+1}_+, \tag{10}$$

then $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^{n+1}_+) = \Theta$.

(ii) *If*

$$\sup_{0<r<\tau} \varphi(x,r)^{-1} = \infty \quad \text{for some } \tau>0 \text{ and for all } x \in \mathbb{R}^{n+1}_+, \tag{11}$$

then $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^{n+1}_+) = \Theta$.

Remark 2.1. *Let Φ be a Young function. By Ω_Φ, we denote the sets of all positive measurable functions φ on $\mathbb{R}^{n+1}_+ \times (0,\infty)$ such that for all $t>0$,*

$$\sup_{x\in\mathbb{R}^n} \left\| \frac{\Phi^{-1}(|\mathcal{E}^+(x,r)|^{-1})}{\varphi(x,r)} \right\|_{L^\infty(t,\infty)} < \infty,$$

and

$$\sup_{x\in\mathbb{R}^n} \left\| \varphi(x,r)^{-1} \right\|_{L^\infty(0,t)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 2.1, we always assume that $\varphi \in \Omega_\Phi$.

Definition 2.3. (parabolic vanishing generalized Orlicz-Morrey Space) The parabolic vanishing generalized Orlicz-Morrey space $VM^{\Phi,\varphi}(\mathbb{R}^{n+1}_+)$ is defined as the space of functions $f \in M^{\Phi,\varphi}(\mathbb{R}^{n+1}_+)$ such that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^{n+1}_+} \frac{1}{\varphi(x,r)} \Phi^{-1}\left(\frac{1}{|\mathcal{E}^+(x,r)|}\right) \|f\|_{L^\Phi(\mathcal{E}^+(x,r))} = 0.$$

Definition 2.4. (parabolic vanishing weak generalized Orlicz-Morrey Space) The parabolic vanishing weak generalized Orlicz-Morrey space $VWM^{\Phi,\varphi}(\mathbb{R}^{n+1}_+)$ is defined as the space of functions

$f \in WM^{\Phi,\varphi}(\mathbb{R}^{n+1}_+)$ such that

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^{n+1}_+}\frac{1}{\varphi(x,r)}\Phi^{-1}\left(\frac{1}{|\mathcal{E}^+(x,r)|}\right)\ \|f\|_{L^\Phi(\mathcal{E}^+(x,r))}=0.$$

The vanishing Morrey space $VM_{p,\lambda}(\mathbb{R}^n)$ of the classical Morrey spaces $M_{p,\lambda}(\mathbb{R}^n)$ was introduced by Vitanza in [49] and applied there to obtain a regularity result for elliptic partial differential equations. Later in [50] Vitanza proved an existence theorem for a Dirichlet problem, under weaker assumptions then those introduced by Miranda in [35], and a $W^{3,2}$ regularity result assuming that the partial derivatives of the coefficients of the highest and lower order terms belong to vanishing Morrey spaces depending on the dimension. Ragusa [46] also proved a sufficient condition for commutators of fractional integral operators to belong to vanishing Morrey spaces $VM^{p,\lambda}(\mathbb{R}^{n+1}_+)$. About commutator operators in vanishing Morrey spaces see the papers [2, 13, 26, 27, 45, 46].

Remark 2.2 *By $\Omega_{\Phi,1}$, we denote the sets of all positive measurable functions φ on $\mathbb{R}^{n+1}_+\times(0,\infty)$ such that*

$$\lim_{r\to 0}\frac{1}{\Phi^{-1}(r^{-n-2})\inf_{x\in\mathbb{R}^{n+1}_+}\varphi(x,r)}=0 \tag{12}$$

and

$$\inf_{x\in\mathbb{R}^{n+1}_+}\inf_{r>\delta}\varphi(x,r)>0,\ \textit{for some } \delta>0. \tag{13}$$

For the non-triviality of the space $VM^{\Phi,\varphi}(\mathbb{R}^{n+1}_+)$ we always assume that $\varphi\in\Omega_{\Phi,1}$.

The spaces $VM^{\Phi,\varphi}(\mathbb{R}^{n+1}_+)$ and $WVM_{\Phi,\varphi}(\mathbb{R}^{n+1}_+)$ are Banach spaces with respect to the norms

$$\|f\|_{VM^{\Phi,\varphi}}\equiv\|f\|_{M^{\Phi,\varphi}}=\sup_{x\in\mathbb{R}^{n+1}_+,r>0}\frac{1}{\varphi(x,r)}\Phi^{-1}\left(\frac{1}{|\mathcal{E}^+(x,r)|}\right)\ \|f\|_{L^\Phi(\mathcal{E}^+(x,r))},$$

$$\|f\|_{VWM^{\Phi,\varphi}}\equiv\|f\|_{WM^{\Phi,\varphi}}=\sup_{x\in\mathbb{R}^{n+1}_+,r>0}\frac{1}{\varphi(x,r)}\Phi^{-1}\left(\frac{1}{|\mathcal{E}^+(x,r)|}\right)\ \|f\|_{WL^\Phi(\mathcal{E}^+(x,r))},$$

respectively. The spaces $VM^{\Phi,\varphi}(\mathbb{R}^{n+1}_+)$ and $VWM^{\Phi,\varphi}(\mathbb{R}^{n+1}_+)$ are closed subspaces of the Banach spaces $M^{\Phi,\varphi}(\mathbb{R}^{n+1}_+)$ and $WM^{\Phi,\varphi}(\mathbb{R}^{n+1}_+)$, respectively, which may be shown by standard means.

3. Parabolic non-singular integral operators in the space $VM^{\Phi,\varphi}(\mathbb{D}^{n+1}_+)$

For any $x=(x',t)=(x'',x_n,t)\in\mathbb{D}^{n+1}_+$ define $\tilde{x}=(x'',-x_n,t)$ and recall that $x^0=(x',0)$. Also define $\mathcal{E}^+_r\equiv\mathcal{E}^+(x^0,r)=\mathcal{E}(x^0,r)\cap\mathbb{D}^{n+1}_+$, $2\mathcal{E}^+_r=\mathcal{E}^+(x^0,2r)$.

For proving our main results, we need the following estimate, which was proved in [12].

Lemma 3.1. *Let $\mathcal{R}$ be a parabolic non-singular integral operator, defined by* (2)*, Φ any Young function, $f\in L^\Phi_{\mathrm{loc}}(\mathbb{D}^{n+1}_+)$, be such that*

$$\int_1^\infty\|f\|_{L^\Phi(\mathcal{E}^+(x^0,t))}\,\Phi^{-1}\big(t^{-n-2}\big)\,\frac{dt}{t}<\infty, \tag{14}$$

i) If $\Phi\in\Delta_2\bigcap\nabla_2$, then

$$\|\mathcal{R}f\|_{L^\Phi(\mathcal{E}^+(x^0,r))}\le\frac{C}{\Phi^{-1}\big(r^{-n-2}\big)}\int_{2r}^\infty\|f\|_{L^\Phi(\mathcal{E}^+(x^0,t))}\,\Phi^{-1}\big(t^{-n-2}\big)\,\frac{dt}{t}. \tag{15}$$

ii) If $\Phi \in \Delta_2$, then

$$\|\mathcal{R}f\|_{WL^\Phi(\mathcal{E}^+(x^0,r))} \leq \frac{C}{\Phi^{-1}\big(r^{-n-2}\big)} \int\limits_{2r}^{\infty} \|f\|_{L^\Phi(\mathcal{E}^+(x^0,t))}\, \Phi^{-1}\big(t^{-n-2}\big)\, \frac{dt}{t}, \tag{16}$$

where the constants are independent of x^0, r and f.

By using Lemma 3.1 the following statement was proved in [43], see also [12].

Theorem 3.1. *Let $\mathcal{R}$ be a parabolic non-singular integral operator, defined by (2), $\Phi \in \Delta_2$ and $\varphi_1, \varphi_2 \in \Omega_\Phi$ satisfy the condition*

$$\int\limits_{r}^{\infty} \left(\operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-n-2}\big)} \right) \Phi^{-1}\big(t^{-n-2}\big) \frac{dt}{t} \leq C\, \varphi_2(x,r), \tag{17}$$

where C does not depend on x and r.
i) Then the operator $\mathcal{R}$ is bounded from $M^{\Phi,\varphi_1}(\mathbb{D}^{n+1}_+)$ to $WM^{\Phi,\varphi_2}(\mathbb{D}^{n+1}_+)$ and

$$\|\mathcal{R}f\|_{M^{\Phi,\varphi_2}(\mathbb{D}^{n+1}_+)} \leq C\|f\|_{WM^{\Phi,\varphi_1}(\mathbb{D}^{n+1}_+)},$$

with constants independent of f.
ii) If $\Phi \in \nabla_2$, then the operator $\mathcal{R}$ is bounded from $M^{\Phi,\varphi_1}(\mathbb{D}^{n+1}_+)$ in $M^{\Phi,\varphi_2}(\mathbb{D}^{n+1}_+)$ and

$$\|\mathcal{R}f\|_{M^{\Phi,\varphi_2}(\mathbb{D}^{n+1}_+)} \leq C\|f\|_{M^{\Phi,\varphi_1}(\mathbb{D}^{n+1}_+)}, \tag{18}$$

with constants independent of f.

Proof of Theorem 1.1. The statement is derived from Theorem 3.1.

So we only have to prove that

$$\lim_{r\to 0} \sup_{x\in\mathbb{D}^{n+1}_+} \frac{1}{\varphi_1(x,r)}\, \Phi^{-1}\left(\frac{1}{|\mathcal{E}^+(x,r)|}\right) \|f\|_{L^\Phi(\mathcal{E}^+(x,r))} = 0,$$

$$\Rightarrow \lim_{r\to 0} \sup_{x\in\mathbb{D}^{n+1}_+} \frac{1}{\varphi_2(x,r)} \Phi^{-1}\left(\frac{1}{|\mathcal{E}^+(x,r)|}\right) \|\mathcal{R}f\|_{L^\Phi(\mathcal{E}^+(x,r))} = 0, \tag{19}$$

and

$$\lim_{r\to 0} \sup_{x\in\mathbb{D}^{n+1}_+} \frac{1}{\varphi_1(x,r)} \Phi^{-1}\left(\frac{1}{|\mathcal{E}^+(x,r)|}\right) \|f\|_{L^\Phi(\mathcal{E}^+(x,r))} = 0,$$

$$\Rightarrow \lim_{r\to 0} \sup_{x\in\mathbb{D}^{n+1}_+} \frac{1}{\varphi_2(x,r)} \Phi^{-1}\left(\frac{1}{|\mathcal{E}^+(x,r)|}\right) \|\mathcal{R}f\|_{WL^\Phi(\mathcal{E}^+(x,r))} = 0. \tag{20}$$

In this estimation, we follow some ideas of [47] in such passage to the limit in the case $\Phi(r) = r^p$, but base ourselves on Lemma 3.1.

To show that $\sup\limits_{x\in\mathbb{D}^{n+1}_+} \frac{1}{\varphi_2(x,r)}\, \Phi^{-1}\left(\frac{1}{|\mathcal{E}^+(x,r)|}\right) \|\mathcal{R}f\|_{L^\Phi(\mathcal{E}^+(x,r))} < \varepsilon$ for small r, we split the right-hand side of (15):

$$\varphi_2(x,r)^{-1}\, \Phi^{-1}\left(\frac{1}{|\mathcal{E}^+(x,r)|}\right) \|\mathcal{R}f\|_{L_\Phi(\mathcal{E}^+(x,r))} \leq C[I_\delta(x,r) + J_\delta(x,r)], \tag{21}$$

where $\delta_0 > 0$ (we may take $\delta_0 < 1$), and

$$I_\delta(x,r) := \frac{1}{\varphi_2(x,r)} \left(\int\limits_{r}^{\delta_0} \frac{\varphi_1(x,t)}{t}\, \varphi_1(x,t)^{-1}\, \|f\|_{L_\Phi(\mathcal{E}^+(x,t))} dt \right),$$

and

$$J_\delta(x,r) := \frac{1}{\varphi_2(x,r)} \left(\int_{\delta_0}^{\infty} \frac{\varphi_1(x,t)}{t} \, \varphi_1(x,t)^{-1} \, \|f\|_{L_\Phi(\mathcal{E}^+(x,t))} dt \right),$$

and it is supposed that $r < \delta_0$. Now we choose any fixed $\delta_0 > 0$ such that

$$\sup_{x \in \mathbb{D}_+^{n+1}} \varphi_1(x,t)^{-1} \, \Phi^{-1} \left(\frac{1}{|\mathcal{E}^+(x,r)|} \right) \, \|f\|_{L^\Phi(\mathcal{E}^+(x,r))} < \frac{\varepsilon}{2CC_0},$$

where C and C_0 are constants from (21) and (4). This allows to estimate the first term uniformly in $r \in (0, \delta_0)$:

$$\sup_{x \in \mathbb{D}_+^{n+1}} C I_{\delta_0}(x,r) < \frac{\varepsilon}{2}, \;\; 0 < r < \delta_0.$$

The estimation of the second term now may be made already by the choice of r sufficiently small. Indeed, thanks to the condition (12) we have

$$J_\delta(x,r) \le c_{\delta_0} \|f\|_{VM^{\Phi,\varphi}} \frac{1}{\varphi(x,r)},$$

where c_{δ_0} is the constant from (3). Then, by (12) it suffices to choose r small enough such that

$$\sup_{x \in \mathbb{D}_+^{n+1}} \frac{1}{\varphi(x,r)} \le \frac{\varepsilon}{2c_{\delta_0} \|f\|_{VM^{\Phi,\varphi}}},$$

which completes the proof of (19).

The proof of (20) is similar to the proof of (19).

□

4. Commutators of non-singular integrals in the space $M^{\Phi,\varphi}(\mathbb{D}_+^{n+1})$

For a function $b \in BMO$ define the commutator $[b, \mathcal{R}]f = b\mathcal{R}f - \mathcal{R}(bf)$. Our aim is to show boundedness of $[b, \mathcal{R}]$ in $M^{\Phi,\varphi}(\mathbb{D}_+^{n+1})$. For this goal, we recall some well-known properties of the BMO functions.

Lemma 4.1. (John-Nirenberg lemma, [31]) *Let $b \in BMO$ and $p \in (1, \infty)$. Then for any ball $\mathcal{E}$ there holds*

$$\left(\frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} |b(y) - b_{\mathcal{E}}|^p dy \right)^{\frac{1}{p}} \le C(p) \|b\|_*. \tag{22}$$

Definition 4.1. *A Young function Φ is said to be of upper type p (resp. lower type p) for some $p \in [0, \infty)$, if there exists a positive constant C such that, for all $t \in [1, \infty)$(resp. $t \in [0, 1]$) and $s \in [0, \infty)$,*

$$\Phi(st) \le C t^p \Phi(s).$$

Remark 4.1 *We know that if Φ is lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$, then $\Phi \in \Delta_2 \cap \nabla_2$. Conversely if $\Phi \in \Delta_2 \cap \nabla_2$, then Φ is lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$ (see* [32]*).*

Before proving the main theorems, we need the following lemma.

Lemma 4.2. [30] *Let $b \in BMO(\mathbb{D}_+^{n+1})$. Then there is a constant $C > 0$ such that*

$$\left| b_{\mathcal{E}_r^+} - b_{\mathcal{E}_t^+} \right| \le C \|b\|_* \ln \frac{t}{r} \quad \text{for} \quad 0 < 2r < t,$$

where C is independent of b, x, r, and t.

In the following lemma which was proved in [23] we provide a generalization of the property (22) from L^p-norms to Orlicz norms.

Lemma 4.3. *Let $b \in BMO(\mathbb{D}_+^{n+1})$ and Φ be a Young function. Let Φ is lower type p_0 and upper type p_1 with $1 \le p_0 \le p_1 < \infty$, then*

$$\|b\|_* \approx \sup_{x \in \mathbb{D}_+^{n+1}, r>0} \Phi^{-1}\big(r^{-n-2}\big) \left\|b(\cdot) - b_{\mathcal{E}^+(x,r)}\right\|_{L^\Phi(\mathcal{E}^+(x,r))}.$$

Remark 4.4. Note that Lemma 4.3 for the variable exponent Lebesgue space $L^{p(\cdot)}$ case was proved in [29].

Definition 4.2 *Let Φ be a Young function. Let*

$$a_\Phi := \inf_{t \in (0,\infty)} \frac{t\Phi'(t)}{\Phi(t)}, \qquad b_\Phi := \sup_{t \in (0,\infty)} \frac{t\Phi'(t)}{\Phi(t)}.$$

Remark 4.5. It is known that $\Phi \in \Delta_2 \cap \nabla_2$ if and only if $1 < a_\Phi \le b_\Phi < \infty$ (See, for example, [33]).

Remark 4.6. Remark 4.5 and Remark 4.1 show us that a Young function Φ is lower type p_0 and upper type p_1 with $1 < p_0 \le p_1 < \infty$ if and only if $1 < a_\Phi \le b_\Phi < \infty$.

To estimate the commutator we shall employ the same idea which we used in the proof of Lemma 3.1 (see [44]).

Lemma 4.4. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$ and $b \in BMO(\mathbb{D}_+^{n+1})$. Suppose that for all $f \in L^\Phi_{\rm loc}(\mathbb{D}_+^{n+1})$ and $r > 0$ holds*

$$\int_1^\infty \Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L^\Phi(\mathcal{E}^+(x^0,t))} \, \Phi^{-1}\big(t^{-n-2}\big) \frac{dt}{t} < \infty. \tag{23}$$

Then

$$\|[b,\mathcal{R}]f\|_{L^\Phi(\mathcal{E}_r^+)} \le \frac{C\|b\|_*}{\Phi^{-1}\big(r^{-n-2}\big)} \int_{2r}^\infty \Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L^\Phi(\mathcal{E}^+(x^0,t))} \, \Phi^{-1}\big(t^{-n-2}\big) \frac{dt}{t}. \tag{24}$$

By using Lemma 4.4 the following statement was proved in [44], see also [12].

Theorem 4.1. *Let $b \in BMO(\mathbb{D}_+^{n+1})$, $\mathcal{R}$ be a parabolic non-singular integral operator, defined by (2), and $\Phi \in \Delta_2 \cap \nabla_2$, $\varphi_1, \varphi_2 \in \Omega_\Phi$ satisfy the condition*

$$\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) \left(\operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-n-2}\big)} \right) \Phi^{-1}\big(t^{-n-2}\big) \frac{dt}{t} \le C\, \varphi_2(x,r), \tag{25}$$

where C does not depend on x and r. Then the operator $[b,\mathcal{R}]$ is bounded from $M^{\Phi,\varphi_1}(\mathbb{D}_+^{n+1})$ to $M^{\Phi,\varphi_2}(\mathbb{D}_+^{n+1})$ and

$$\|[b,\mathcal{R}]f\|_{M^{\Phi,\varphi_2}(\mathbb{D}_+^{n+1})} \le C\|b\|_* \, \|f\|_{M^{\Phi,\varphi_1}(\mathbb{D}_+^{n+1})}, \tag{26}$$

with a constant independent of f.

Proof of Theorem 1.2. The proof follows more or less the same lines as for Theorem 3.1, but now the arguments are different due to the necessity to introduce the logarithmic factor into the assumptions.

The norm inequality having already been provided by Theorem 4.1, we only have to prove the implication

$$\lim_{r\to 0}\sup_{x\in\mathbb{D}_+^{n+1}}\frac{1}{\varphi_1(x,r)}\Phi^{-1}\left(\frac{1}{|\mathcal{E}^+(x,r)|}\right)\|f\|_{L^\Phi(\mathcal{E}^+(x,r))}=0$$
$$\Longrightarrow\ \frac{1}{\varphi_2(x,r)}\Phi^{-1}\left(\frac{1}{|\mathcal{E}^+(x,r)|}\right)\|[b,\mathcal{R}]f\|_{L^\Phi(\mathcal{E}^+(x,r))}=0. \tag{27}$$

To check that

$$\sup_{x\in\mathbb{D}_+^{n+1}}\frac{1}{\varphi_2(x,r)}\Phi^{-1}\left(\frac{1}{|\mathcal{E}^+(x,r)|}\right)\|[b,\mathcal{R}]f\|_{L^\Phi(\mathcal{E}^+(x,r))}<\varepsilon\quad\text{for small}\ \ r,$$

we use the estimate (24):

$$\frac{1}{\varphi_2(x,r)}\Phi^{-1}\left(\frac{1}{|\mathcal{E}^+(x,r)|}\right)\|[b,\mathcal{R}]f\|_{L^\Phi(\mathcal{E}^+(x,r))}\lesssim\frac{\|b\|_*}{\varphi_2(x,r)}\int_r^\infty\left(1+\ln\frac{t}{r}\right)\|f\|_{L_\Phi(B(x_0,t))}\frac{dt}{t}.$$

We take $r<\delta_0$ where δ_0 will be chosen small enough and split the integration:

$$\frac{1}{\varphi_2(x,r)}\Phi^{-1}\left(\frac{1}{|\mathcal{E}^+(x,r)|}\right)\|[b,\mathcal{R}]f\|_{L^\Phi(\mathcal{E}^+(x,r))}\le C[I_{\delta_0}(x,r)+J_{\delta_0}(x,r)], \tag{28}$$

where

$$I_{\delta_0}(x,r):=\frac{1}{\varphi_2(x,r)}\int_r^{\delta_0}\left(1+\ln\frac{t}{r}\right)\|f\|_{L^\Phi(\mathcal{E}^+(x,r))}\frac{dt}{t}$$

and

$$J_{\delta_0}(x,r):=\frac{1}{\varphi_2(x,r)}\int_{\delta_0}^\infty\left(1+\ln\frac{t}{r}\right)\|f\|_{L^\Phi(\mathcal{E}^+(x,r))}\frac{dt}{t}.$$

We choose a fixed $\delta_0>0$ such that

$$\sup_{x\in\mathbb{D}_+^{n+1}}\frac{1}{\varphi_1(x,r)}\Phi^{-1}\left(\frac{1}{|\mathcal{E}^+(x,r)|}\right)\|f\|_{L^\Phi(\mathcal{E}^+(x,r))}<\frac{\varepsilon}{2CC_0},\quad t\le\delta_0,$$

where C and C_0 are constants from (28) and (5), which yields, the estimate of the first term uniform in $r\in(0,\delta_0):\ \sup\limits_{x\in\mathbb{D}_+^{n+1}}CI_{\delta_0}(x,r)<\frac{\varepsilon}{2},\ 0<r<\delta_0$.

For the second term, writing $1+\ln\frac{t}{r}\le 1+|\ln t|+\ln\frac{1}{r}$, we obtain

$$J_{\delta_0}(x,r)\le\frac{c_{\delta_0}+\widetilde{c_{\delta_0}}\ \ln\frac{1}{r}}{\varphi_2(x,r)}\|f\|_{M^{\Phi,\varphi}},$$

where c_{δ_0} is the constant from (7) with $\delta=\delta_0$ and $\widetilde{c_{\delta_0}}$ is a similar constant with omitted logarithmic factor in the integrand. Then, by (6) we can choose small r such that $\sup_{x\in\mathbb{D}_+^{n+1}}J_{\delta_0}(x,r)<\frac{\varepsilon}{2}$, which completes the proof.

□

5. Conclusion

In this paper, we obtain the sufficient conditions on general Young function Φ and functions φ_1, φ_2 which ensure the boundedness of parabolic non-singular integral operator $\mathcal{R}$ from one parabolic vanishing generalized Orlicz-Morrey spaces $VM^{\Phi,\varphi_1}(\mathbb{D}_+^{n+1})$ to another $VM^{\Phi,\varphi_2}(\mathbb{D}_+^{n+1})$, from $VM^{\Phi,\varphi_1}(\mathbb{D}_+^{n+1})$ to parabolic vanishing weak generalized Orlicz-Morrey spaces $VWM^{\Phi,\varphi_2}(\mathbb{D}_+^{n+1})$ and the boundedness of commutator of the parabolic non-singular integral operator $[b,\mathcal{R}]$ from $VM^{\Phi,\varphi_1}(\mathbb{D}_+^{n+1})$ to $VM^{\Phi,\varphi_2}(\mathbb{D}_+^{n+1})$.

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