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FORMULA FOR THE GRADIENT IN THE OPTIMAL CONTROL PROBLEM FOR THE NON-LINEAR SYSTEM OF THE HYPERBOLIC EQUATIONS WITH NON-LOCAL BOUNDARY CONDITIONS

H.F. GULIYEV[1], Y.A. SHARIFOV[2]

Abstract. In this work the optimal control problem for the nonlinear system of the first order hyperbolic equations with non-local conditions is investigated. The existence and uniqueness of the solutions of the boundary value problem under some conditions have been proved and the formula for gradient of the functional is derived.

Keywords: non-local conditions, existence, uniqueness, optimal control, gradient formula.

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1. Introduction

Partial differential equations with non-local boundary conditions are intensively studied last years. Nonlocal problems are called the problems which include some relations between the values of the sought functions on the boundary and inside of the domain instead of traditional boundary conditions. Often as such relations are taken the conditions, involving different values of the sought solution.

In the books [11], [12] various examples have been considered from biology, sociology, techniques, when mathematical models are described by the hyperbolic equations with nonlocal conditions. In [13] linear hyperbolic system is considered with integral and multipoint boundary conditions and necessary and sufficient conditions of solvability are proved. One dimensional nonlinear hyperbolic equation with integral and multipoint is studied by [8].In this work boundary conditions are given on the characteristics of the equation. The existence and uniqueness of the classical solution is also proved. Linear and quasilinear hyperbolic equations are considered with integral and multipoint boundary conditions and the theorems of existence and uniqueness of the classical solution have been proved in [1], [4], [15]-[18]. In [2], [3], [10] different hyperbolic equations with various nonlocal boundary conditions are investigated.

So, investigation of the optimal control problems for such processes is an actual and natural. But such problems are not studied enough. Some optimal control problems for the second order hyperbolic systems with different nonlocal conditions are considered in [5]-[7], [19], [20]. Optimal control problem for hyperbolic systems occur in the heterogenic reactors and in the chemical processes [9], [14], [21].

[1] Baku State University, Baku, Azerbaijan,
e-mail: hkuliyev@rambler.ru
[2] Institute of Cybernetics of ANAS, Baku, Azerbaijan,
e-mail: sharifov22@rambler.ru
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2. Problem statement

In this paper, we consider optimal control problem for the nonlinear system of the first order hyperbolic equations in the bounded square. In this present study, the boundary conditions are given on the characteristics of the hyperbolic system by the help of two-point conditions. It is to be noted that, this problem is considered first time in the literature.

Let the functional

$$J(u) = \int_0^T P(t, y(t,0)y(t,l))dt + \int_0^l Q(s, x(0,s), x(T,s))ds \tag{1}$$

should be minimized under the conditions

$$\begin{cases} \frac{\partial x(t,s)}{\partial t} = f(t, s, x(t,s), y(t,s), u(t,s)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (t,s) \in [0,T] \times [0,l] = G, \\ \frac{\partial y(t,s)}{\partial s} = g(t, s, x(t,s), y(t,s), u(t,s)), \end{cases} \tag{2}$$

$$\begin{aligned} &x(0,s) + \alpha(s)x(T,s) = \varphi(s), \;\; 0 \le s \le l, \\ &y(t,0) + \beta(t)y(t,l) = \psi(t), \;\; 0 \le t \le T, \end{aligned} \tag{3}$$

$$u = u(t,s) \in U \subseteq L_2^r(G), \tag{4}$$

where T, l are given positive numbers, $(x, y) = (x_1, ..., x_n; y_1, ..., y_m)$ are phase variables, $u = (u_1, ..., u_r)$ is controlling influence, $f = (f_1, ..., f_n)$, $g = (g_1, ..., g_m)$, $\varphi = (\varphi_1, ..., \varphi_n)$, $\psi = (\psi_1, ..., \psi_m)$ are given vector-functions, $\alpha(s) \in R^{n\times n}, \beta(t) \in R^{m\times m}$ are matrix-functions, $P(t, a, b)$, $Q(s, c, d)$- given functions. As a solution of the problem (2), (3) corresponding to the control $u = u(t,s) \in L_2^r(G)$, we take the vector-function $z(t,s) = (x(t,s), y(t,s)) \in L_2^{n+m}(G)$,having generalized derivatives $\frac{\partial x_i(t,s)}{\partial t} \in L_2(G)$,$i = 1, ..., n$, $\frac{\partial y_j(t,s)}{\partial s}$, $j = 1, ..., m$, satisfying to the equations (2) almost everywhere in G, and the conditions (3) in the sense of equality of the corresponding traces of the functions $x(t,s)$, $y(t,s)$. As a class admissible controls we take the given set from $L_2^r(G)$. Suppose that the following conditions are satisfied:

(1) The functions $f_i(t, s, x, y, u)$, $i = \overline{1,n}$, $g_j(t, s, x, y, u)$, $j = \overline{1,m}$ together with their partial derivatives f_{ix}, f_{iy}, f_{iu}, g_{jx}, g_{jy}, g_{ju} are continuous over the set of variables $(t, s, x, y, u) \in [0,T]\times[0,l]\times R^n\times R^m\times R^r$ and satisfy to the Lipchitz condition over the variables(x, y, u), i.e.

$$|f(t, s, x + \bar{x}, y + \bar{y}, u + \bar{u}) - f(t, s, x, y, u)| \le K_1^{(1)} |\bar{x}| + K_2^{(1)} |\bar{y}| + K_3^{(1)} |\bar{u}|,$$

$$|g(t, s, x + \bar{x}, y + \bar{y}, u + \bar{u}) - g(t, s, x, y, u)| \le K_1^{(2)} |\bar{x}| + K_2^{(2)} |\bar{y}| + K_3^{(2)} |\bar{u}|;$$

where $K_i^{(j)}$ are Lipchitz constants, $i = 1, 2, 3; j = 1, 2$.

(2) $\alpha(s)$ is $n \times n$ a matrix with measurable bounded elements in the interval $[0, l]$, $\beta(t)$ is $m\times m$ matrix with measurable bounded elements in the interval $[0,T]$, $\|\alpha\| < 1$, $\|\beta\| < 1$, where $\|\alpha\|$, $\|\beta\|$ are norms of the matrices $\alpha(s)$ and $\beta(t)$ correspondingly, $\varphi(s)$ is given vector function from $L_2^n(0,l)$, $\psi(t)-$ given vector function from $L_2^m(0,l)$;

(3) Functions $P(t, a, b)$ and $Q(s, c, d)$ are continuous over the set of variables $(t, a, b) \in [0,T] \times R^m \times R^m$ and $(s, c, d) \in [0,l] \times R^n \times R^n$ together with their partial derivatives P_a, P_b,Q_c, Q_d and satisfy to Lipchitz condition over the variables (a, b) and (c, d) correspondingly.

3. Reduction of the boundary problem (2),(3) to the system of integral equations

Now, let's reduce the problem (2),(3) to the equivalent system of the integral equations. It is clear that

$$\int_0^t \frac{\partial x(\tau,s)}{\partial \tau} d\tau = \int_0^t f(\tau,s,x(\tau,s),y(\tau,s),u(\tau,s))d\tau.$$

From this we have

$$x(t,s) - x(0,s) = \int_0^t f(\tau,s,x(\tau,s),y(\tau,s),u(\tau,s))d\tau,$$

or

$$x(t,s) = l(s) + \int_0^t f(\tau,s,x(\tau,s),y(\tau,s),u(\tau,s))d\tau, \tag{5}$$

where $l(s)$ is unknown yet function. We assume that the function $x(t,s)$ from (5) satisfies to first condition from (3), i.e.

$$l(s) + \alpha(s)\left[l(s) + \int_0^T f(t,s,x(t,s),y(t,s),u(t,s))dt\right] = \varphi(s).$$

Thus

$$[E+\alpha(s)]\, l(s) = \varphi(s) - \alpha(s)\int_0^T f(t,s,x(t,s),y(t,s)u(t,s))dt,$$

where E is unit matrix. As $\|\alpha\| < 1$, $[E+\alpha(s)]^{-1}$ exists and we can define the function $l(s)$:

$$l(s) = [E+\alpha(s)]^{-1}\varphi(s) - [E+\alpha(s)]^{-1}\alpha(s)\int_0^T f(t,s,x(t,s),y(t,s),u(t,s))dt. \tag{6}$$

Let substitute $l\,(s)$ into the equal it (5). Then

$$x(t,s) = [E+\alpha(s)]^{-1}\varphi(s) - [E+\alpha(s)]^{-1}\alpha(s)\int_0^T f(t,s,x(t,s),y(t,s),u(t,s))dt +$$

$$+\int_0^t f(\tau,s,x(\tau,s),y(\tau,s),u(\tau,s))d\tau.$$

The above equality can be rewritten as

$$\begin{aligned} x(t,s) &= [E+\alpha(s)]^{-1}\varphi(s) + \int_0^t \left\{E - [E+\alpha(s)]^{-1}\alpha(s)\right\}\times \\ &\times f(\tau,s,x(\tau,s),y(\tau,s),u(\tau,s))d\tau - \int_t^T (E+\alpha(s))^{-1}\alpha(s) f(\tau,s,x(\tau,s),y(\tau,s),u(\tau,s))d\tau. \end{aligned} \tag{7}$$

Let $E-(E+\alpha(s))^{-1}\alpha(s)=(E+\alpha(s))^{-1}$ and introduce the matrix function

$$M(t,\tau,s)=\begin{cases}(E+\alpha(s))^{-1}, & 0\le\tau\le t,\\ \\ -(E+\alpha(s))^{-1}\alpha(s), & t\le\tau\le T,\end{cases}$$

Therefore, (7) may be transferred as

$$x(t,s)=(E+\alpha(s))^{-1}\varphi(s)+\int\limits_0^T M(t,\tau,s)f(\tau,s,x(\tau,s),y(\tau,s),u(\tau,s))d\tau.$$

Similarly, for the function $y\,(t,s)$ we obtain

$$y(t,s)=(E+\beta(t))^{-1}\psi(t)+\int\limits_0^l N(t,s,\eta)g(t,\eta,x(t,\eta),y(t,\eta),u(t,\eta))d\eta,$$

where

$$N(t,s,\eta)=\begin{cases}(E+\beta(t))^{-1}, & 0\le\eta\le s,\\ \\ -(E+\beta(t))^{-1}\beta(t), & s\le\eta\le l.\end{cases}$$

Hence, the problem (2), (3) is equivalent to the following system of integral equations

$$\begin{aligned}x(t,s)&=(E+\alpha(s))^{-1}\varphi(s)+\int\limits_0^T M(t,\tau,s)f(\tau,s,x(\tau,s),y(\tau,s),u(\tau,s))d\tau,\\ y(t,s)&=(E+\beta(t))^{-1}\psi(t)+\int\limits_0^l N(t,s,\eta)g(t,\eta,x(t,\eta),y(t,\eta),u(t,\eta))d\eta.\end{aligned}\tag{8}$$

4. Existence of the solution of the boundary problem (2), (3).

It may be shown that under the put conditions the system of integral equations (8) has unique solution $(x(t,s),y(t,s))$ from $L_2^n(G)\times L_2^m(G)$ and this solution has properties $\frac{\partial x(t,s)}{\partial t}\in L_2^n(G)$, $\frac{\partial y(t,s)}{\partial s}\in L_2^m(G)$. From the conditions on $\alpha\,(s)\,,\beta\,(t)$ follows that

$$\|M(t,\tau,s)\|\le\frac{1}{1-\|\alpha\|}\equiv M,\ \ \|N(t,s,\eta)\|\le\frac{1}{1-\|\beta\|}\equiv N.$$

Theorem 4.1. *Let the conditions 1)-3) are satisfied. Moreover, the maximal eigenvalue of matrix*

$$\begin{pmatrix}MK_1^{(1)}T & MK_2^{(1)}l\\ \\ NK_1^{(2)}T & NK_2^{(2)}l\end{pmatrix}$$

is less than 1. Then the system of the integral equations (8) *has the uniqueness solutions* $(x(t,s),y(t,s))$ *from* $L_2^n(G)\times L_2^m(G)$.

Proof. To prove we introduce the operator

$$(F(x,y))(t,s)=\begin{pmatrix} (E+\alpha(s))^{-1}\varphi(s)+\int\limits_0^T M(t,\tau,s)f(\tau,s,x(\tau,s),y(\tau,s),u(\tau,s))d\tau \\ \\ (E+\beta(t))^{-1}\psi(t)+\int\limits_0^l N(t,s,\eta)g(t,\eta,x(t,\eta),y(t,\eta),u(t,\eta))d\eta \end{pmatrix}.$$

It is shown that this operator under the above conditions is contraction. □

From this theorem follows:

Corollary 4.1. *Let the conditions 1)-3) are satisfied. Moreover,*

$$MK_1T+NK_2l<1,$$

where $K_1=\max\left\{K_1^{(1)},K_2^{(1)}\right\}$, $K_2=\max\left\{K_1^{(2)},K_2^{(2)}\right\}$.

Then the system of integral equations (8) has unique solution $(x(t,s),y(t,s))$ from $L_2^n(G)\times L_2^m(G)$.

5. The gradient of the problem (1)-(4)

Let $u,\ u+\bar u\in U$ be two different controls, (x,y), $(x+\bar x,\ y+\bar y)$ -corresponding them solutions of (8). Then

$$\begin{aligned} &\bar x(t,s)=\int\limits_0^T M(t,\tau,s)\left[f(\tau,s,x(\tau,s)+\bar x(\tau,s),y(\tau,s)+\bar y(\tau,s),u(\tau,s)+\bar u(\tau,s))-\right.\\ &\left.-f(\tau,s,x(\tau,s),y(\tau,s),u(\tau,s))\right]d\tau,\\ &\bar y(t,s)=\int\limits_0^l N(t,s,\eta)\left[g(t,\eta,x(t,\eta)+\bar x(t,\eta),y(t,\eta)+\bar y(t,\eta),u(t,\eta)+\bar u(t,\eta))-\right.\\ &\left.-g(t,\eta,x(t,\eta),y(t,\eta),u(t,\eta))\right]d\eta \end{aligned}\tag{9}$$

As the functions $f$and g satisfy to the Lipchitz condition over the variables (x,y,u) from (9) we get

$$\begin{aligned} &|\bar x(t,s)|\le M\int\limits_0^T\left[K_1^{(1)}|\bar x(t,s)|+K_2^{(1)}|\bar y(t,s)|+K_3^{(1)}|\bar u(t,s)|\right]dt,\\ &|\bar y(t,s)|\le N\int\limits_0^l\left[K_1^{(2)}|\bar x(t,s)|+K_2^{(2)}|\bar y(t,s)|+K_3^{(2)}|\bar u(t,s)|\right]ds, \end{aligned}\tag{10}$$

here $|\cdot|$ is a norm of the vector. From (10) follows that

$$\begin{gathered}
\underset{0\le t\le T}{vrai\max}\int\limits_0^l |\bar{x}(t,s)|\,ds \le MK_1^{(1)}T\,\underset{0\le t\le T}{vrai\max}\int\limits_0^l |\bar{x}(t,s)|\,ds+ \\
+MK_2^{(1)}l\,\underset{0\le s\le l}{vrai\max}\int\limits_0^T |\bar{y}(t,s)|\,ds+MK_3^{(1)}\int\limits_0^T\int\limits_0^l |\bar{u}|\,dtds, \\
\underset{0\le s\le l}{vrai\max}\int\limits_0^T |\bar{y}(t,s)|\,dt \le NK_1^{(2)}T\,\underset{0\le t\le T}{vrai\max}\int\limits_0^l |\bar{x}(t,s)|\,ds+ \\
+NK_2^{(2)}l\,\underset{0\le s\le l}{vrai\max}\int\limits_0^T |\bar{y}(t,s)|\,dt+NK_3^{(2)}\int\limits_0^T\int\limits_0^l |\bar{u}|\,dtds.
\end{gathered} \tag{11}$$

If define $z=\begin{pmatrix} \underset{0\le t\le T}{vrai\max}\int\limits_0^l |\bar{x}(t,s)|\,ds \\ \underset{0\le s\le l}{vrai\max}\int\limits_0^T |\bar{y}(t,s)|\,dt \end{pmatrix}$, then from (11) we get

$$z\le\begin{pmatrix} MK_1^{(1)}T & MK_2^{(1)}l \\ NK_1^{(2)}T & NK_2^{(2)}l \end{pmatrix}z+\begin{pmatrix} MK_3^{(1)}\sqrt{Tl} \\ NK_3^{(2)}\sqrt{Tl} \end{pmatrix}\|\bar{u}\|_{L_2^r(G)}.$$

From this obtained

$$z\le\begin{pmatrix} 1-MK_1^{(1)}T & -MK_2^{(1)}l \\ -NK_1^{(2)}T & 1-NK_2^{(2)}l \end{pmatrix}^{-1}\begin{pmatrix} MK_3^{(1)}\sqrt{Tl} \\ NK_3^{(2)}\sqrt{Tl} \end{pmatrix}\|\bar{u}\|_{L_2^r(G)}. \tag{12}$$

Now we show that the functional (1) is differentiable on $L_2^r(G)$ and its increment may be written as

$$\begin{gathered}
\Delta J(u)=J(u+\bar{u})-J(u)=\int\limits_0^T[P(t,y(t,0)+\bar{y}(t,0),y(t,l)+\bar{y}(t,l))- \\
-P(t,y(t,0),y(t,l))]dt+\int\limits_0^l[Q(s,x(0,s)+\bar{x}(0,s),x(T,s)+\bar{x}(T,s))- \\
-Q(s,x(0,s),x(T,s))]ds.
\end{gathered} \tag{13}$$

From (2) and (3) one may get

$$\frac{\partial\bar{x}(t,s)}{\partial t}-\Delta f(t,s)=0, \tag{14}$$

$$\frac{\partial\bar{y}(t,s)}{\partial t}-\Delta g(t,s)=0, \tag{15}$$

$$\bar{x}(0,s)+\alpha(s)\bar{x}(T,s)=0, \tag{16}$$

$$\bar{y}(t,0)+\beta(t)\bar{y}(t,l)=0, \tag{17}$$

where

$$\Delta f\,(t,s) \equiv f(t,s,x(t,s)+\bar{x}(t,s),y(t,s)+\bar{y}(t,s),u(t,s)+\bar{u}(t,s))-$$
$$-f(t,s,x(t,s),y(t,s),u(t,s)),$$

$$\Delta g\,(t,s) \equiv g(t,s,x(t,s)+\bar{x}(t,s),y(t,s)+\bar{y}(t,s),u(t,s)+\bar{u}(t,s))-$$
$$-g(t,s,x(t,s),y(t,s),u(t,s)).$$

If scalarly multiply both sides of the equation (14) by arbitrary yet vector function $\psi^{(1)}(t,s)$, both sides of (15) by $\psi^{(2)}(t,s)$, equality (16) by $\lambda\,(s)$, both sides of the (17) by and integrate them on the domain G and intervals and correspondingly, then add the obtained equality by (13) may get

$$\begin{aligned}
\Delta J(u) = &\int_0^T \left[P(t,y(t,0)+\bar{y}(t,0),y(t,l)+\bar{y}(t,l)) \;-P(t,y(t,0),y(t,l))\right]dt+\\
&+\int_0^l \left[Q(s,x(0,s)+\bar{x}(0,s),x(T,s)+\bar{x}(T,s)) \;-Q(s,x(0,s),x(T,s))\right]ds+\\
&+\iint_G \left\langle \psi^{(1)}(t,s),-\frac{\partial \bar{x}(t,s)}{\partial t}+\Delta f(t,s)\right\rangle dtds+\\
&+\iint_G \left\langle \psi^{(2)}(t,s),-\frac{\partial \bar{y}(t,s)}{\partial s}+\Delta g(t,s)\right\rangle dtds+\\
&+\int_0^l \langle\lambda(s),\,\bar{x}(0,s)+\alpha(s)\bar{x}(T,s)\rangle\,ds+\int_0^T \langle\mu(t),\bar{y}(t,0)+\beta(t)\bar{y}(t,l)\rangle\,dt,
\end{aligned} \tag{18}$$

where $\langle .,.\rangle$ means scalar product in the Euclidian space.

Define $H(t,s,x,y,\psi^{(1)},\psi^{(2)},u)=\left\langle\psi^{(1)},f(t,s,x,y,u)\right\rangle+\left\langle\psi^{(2)},g(t,s,x,y,u)\right\rangle$. Then the formula (18) for the increment of the functional (1) takes the form:

$$\begin{aligned}
\Delta J(u) = &\int_0^T \left[\langle P_a(t,y(t,0),y(t,l)),\bar{y}(t,0)\rangle+\langle P_b(t,y(t,0)\;,\;y(t,l)),\bar{y}(t,l)\rangle\right]dt+\\
&+\int_0^l \left[\langle Q_c(s,x(0,s),x(T,s)),\bar{x}(0,s)\rangle+\;Q_d(s,x(0,s),x(T,s)),\bar{x}(T,s)\rangle\right]ds+\\
&+\iint_G \left[H(t,s,x+\bar{x},y+\bar{y},\psi^{(1)},\psi^{(2)},u+\bar{u})-H(t,s,x,y,\psi^{(1)},\psi^{(2)},u)\right]dtdx+\\
&+\iint_G \left\langle \frac{\partial \psi^{(1)}(t,s)}{\partial t},\bar{x}(t,s)\right\rangle dtds+\iint_G \left\langle \frac{\partial \psi^{(2)}(t,s)}{\partial s},\bar{y}(t,s)\right\rangle dtds+\\
&+\int_0^l \left\langle\psi^{(1)}(0,s),\bar{x}(0,s)\right\rangle-\left\langle\psi^{(1)}(T,s),\bar{x}(T,s)\right\rangle\Big]\,ds+\\
&+\int_0^T \left[\left\langle\psi^{(2)}(t,0),\bar{y}(t,0)\right\rangle-\left\langle\psi^{(2)}(t,l),\bar{y}(t,l)\right\rangle\right]dt+\\
&+\int_0^l \langle\lambda(s),\bar{x}(0,s)+\alpha(s)\bar{x}(T,s)\rangle\,ds+\int_0^T \langle\mu(t),\bar{y}(t,0)+\beta(t)\bar{y}(t,l)\rangle\,dt+R_1+R_2,
\end{aligned} \tag{19}$$

where

$$R_1 = \int_0^T \langle P_a(t, y(t,0) + \theta_1 \bar{y}(t,0), y(t,l) + \theta_1 \bar{y}(t,l)) - P_a(t, y(t,0), y(t,l)), \bar{y}(t,0)\rangle\, dt +$$
$$+ \int_0^T \langle P_b(t, y(t,0) + \theta_2 \bar{y}(t,0), y(t,l) + \theta_2 \bar{y}(t,l)) - P_b(t, y(t,0), y(t,l)), \bar{y}(t,l)\rangle\, dt,$$
$$R_2 = \int_0^l \langle Q_c(s, x(0,s) + \theta_3 \bar{x}(0,s), x(T,s) + \theta_3 \bar{x}(T,s)) - Q_c(s, x(0,s), x(T,s)), \bar{x}(0,s)\rangle\, ds +$$
$$+ \int_0^l \langle Q_d(s, x(0,s) + \theta_4 \bar{x}(0,s), x(T,s) + \theta_4 \bar{x}(T,s)) - Q_d(s, x(0,s), x(T,s)), \bar{x}(T,s)\rangle\, ds,$$
$$0 < \theta_i < 1, i = \overline{1,4.}$$

If consider

$$\Delta H = H(t,s,x+\bar{x}, y+\bar{y}, \psi^{(1)}, \psi^{(2)}, u+\bar{u}) - H(t,s,x,y,\psi^{(1)},\psi^{(2)},u) =$$
$$= \left\langle H_x(t,s,x,y,\psi^{(1)},\psi^{(2)},u), \bar{x}\right\rangle + \left\langle H_y(t,s,x,y,\psi^{(1)},\psi^{(2)},u), \bar{y}\right\rangle +$$
$$+ \left\langle H_u(t,s,y,\psi^{(1)},\psi^{(2)},u), \bar{u}\right\rangle + \tilde{R}_3 + \tilde{R}_4 + \tilde{R}_5,$$

where

$$\tilde{R}_3 = \left\langle H_x(t,s,x+\theta_5\bar{x}, y+\theta_5\bar{y}, \psi^{(1)}, \psi^{(2)}, u+\theta_5\bar{u}) - H_x(t,s,x,y,\psi^{(1)},\psi^{(2)},u), \Delta x(t,s)\right\rangle,$$
$$\tilde{R}_4 = \left\langle H_y(t,s,x+\theta_6\bar{x}, y+\theta_6\bar{y}, \psi^{(1)}, \psi^{(2)}, u+\theta_6\bar{u}) - H_y(t,s,x,y,\psi^{(1)},\psi^{(2)},u), \Delta y(t,s)\right\rangle,$$
$$\tilde{R}_5 = \left\langle H_u(t,s,x+\theta_7\bar{x}, y+\theta_7\bar{y}, \psi^{(1)}, \psi^{(2)}, u+\theta_7\bar{u}) - H_u(t,s,x,y,\psi^{(1)},\psi^{(2)},u), \bar{u}(t,s)\right\rangle,$$
$$0 < \theta_i < 1, i = \overline{5,7.}$$

Then the equality (19) may be rewritten as

$$\begin{aligned}
\Delta J(u) = &\int_0^T \Big[\Big\langle P_a(t, y(t,0), y(t,l)) + \psi^{(2)}(t,0) + \mu(t), \bar{y}(t,0)\Big\rangle + \\
&+ \langle P_b(t, y(t,0)\ ,\ y(t,l)) + \psi^{(2)}(t,l) + \beta'(t)\mu(t), \bar{y}(t,l)\rangle\Big]\, dt + \\
&+ \int_0^l \Big[\Big\langle Q_c(s, x(0,s), x(T,s)) + \psi^{(1)}(0,s) + \lambda(s), \bar{x}(0,s)\Big\rangle + \\
&+ \left\langle Q_d(s, x(0,s), x(T,s)) + \psi^{(1)}(T,s) + \alpha'(s)\lambda(s), \bar{x}(T,s)\right\rangle\Big]\, ds + \\
&+ \iint_G \Big\langle H_u(t,s,x(t,s),y(t,s),\psi^{(1)}(t,s),\psi^{(2)}(t,s),u(t,s)), \bar{u}(t,s)\Big\rangle\, dtds + \\
&+ \iint_G \Big\langle \tfrac{\partial \psi^{(1)}(t,s)}{\partial t} + \tfrac{\partial H(t,s,x(t,s),y(t,s),\psi^{(1)}(t,s),\psi^{(2)}(t,s),u(t,s))}{\partial x}, \bar{x}(t,s)\Big\rangle\, dtds + \\
&+ \iint_G \Big\langle \tfrac{\partial \psi^{(2)}(t,s)}{\partial s} + \tfrac{\partial H(t,s,x(t,s),y(t,s),\psi^{(1)}(t,s),\psi^{(2)}(t,s),u(t,s))}{\partial y}, \bar{y}(t,s)\Big\rangle\, dtds + R
\end{aligned} \tag{20}$$

where $R=\sum\limits_{i=1}^{5}R_i$; R_1,R_2 is defined above and $R_3=\iint\limits_G \tilde{R}_3 dtds$, $R_4=\iint\limits_G \tilde{R}_4 dtds$, $R_5=\iint\limits_G \tilde{R}_5 dtds$. Using the large choice of $\bar{x}(t,s)$ and $\bar{y}(t,s)$ one can make zero the coefficients of the variables $\bar{y}(t,0)$, $\bar{y}(t,l)$, $\bar{x}(0,s)$, $\bar{x}(T,s),\bar{x}(t,s),\bar{y}(t,s)$ and arrive to the following conjugate problem for the function $\psi^{(1)}(t,s),\psi^{(2)}(t,s)$:

$$\frac{\partial\psi^{(1)}(t,s)}{\partial t}=-\frac{\partial H(t,s,x,y,\psi^{(1)},\psi^{(2)},u)}{\partial x}, \tag{21}$$

$$\frac{\partial\psi^{(2)}(t,s)}{\partial s}=-\frac{\partial H(t,s,x,y,\psi^{(1)},\psi^{(2)},u)}{\partial y},\ (t,s)\in G, \tag{22}$$

$$P_a(t,y(t,0),y(t,l))+\psi^{(2)}(t,0)+\mu(t)=0, \tag{23}$$

$$P_b(t,y(t,0),y(t,l))+\psi^{(2)}(t,l)+\beta'(t)\mu(t)=0,\ \ t\in[0,T]\ , \tag{24}$$

$$Q_c(s,x(0,s),x(T,s))+\psi^{(1)}(0,s)+\lambda(s)=0,\ s\in[0,l] \tag{25}$$

$$Q_d(s,x(0,s),x(T,s))+\psi^{(1)}(T,s)+\alpha'(s)\lambda(s)=0, \tag{26}$$

The problem (21)-(26) is called a parametrical form of the conjugate system. Here the sought functions indeed are $\left\{\psi^{(1)}(t,s),\psi^{(2)}(t,s),\lambda(s),\mu(t)\right\}$. From the systems (23), (24) and (25), (26) one may except the functions $\lambda\left(s\right)$ and $\mu\left(t\right)$, i.e.

$$P_b(t,y(t,0),y(t,l))+\psi^{(2)}(t,l)=\beta'(t)P_a(t,y(t,0),y(t,l))+\beta'(t)\psi^{(2)}(t,0), \tag{27}$$

$$Q_d(s,x(0,s),x(T,s))+\psi^{(1)}(T,s)=\alpha'(s)Q_c(s,x(0,s),x(T,s))+\alpha'(s)\psi^{(1)}(0,s). \tag{28}$$

Thus, for the increment of the functional the following expression is obtained

$$\Delta J(u)=\iint\limits_G\left\langle H_u(t,s,x(t,s),y(t,s),\psi^{(1)}(t,s),\psi^{(2)}(t,s),u(t,s)),\bar{u}(t,s)\right\rangle dtds+R. \tag{29}$$

Under the assumptions above on the functions f_{ix}, f_{iy}, f_{iu}, g_{jx},g_{jy},g_{ju} for the remainder term R of the formula (29) the estimation

$$|R|\le C_0\left\|\bar{u}\right\|^2, C_0=const\ge 0,$$

is true, that follows from (12). From this it follows that the functional (1) is differentiable on $L_2^r\left(G\right)$ and the formula

$$J'(u)=H_u(t,s,x(t,s),y(t,s),\psi^{(1)}(t,s),\psi^{(2)}(t,s),u(t,s))$$

is valid for its gradient.

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Hamlet Guliev was born in 1951 in Jebrail region of Azerbaijan. He graduated from mechanical-mathematical faculty of Baku State University in 1973. He got his Ph.D. degree in 1980 and Doctor of Sciences degree in 1997. Since 2001 he is a Professor at the Department "Mathematical methods of control theory" of Baku State University. His research interests include optimal control problems for the partial differential equations, variation calculus.

Yagub Sharifov was born in 1958 in Gubadli region (Basharat village) of Azerbaijan (it is now under the Armenian occupation). He graduated from mechanical-mathematical faculty of Baku State University in 1980. He got his Ph.D. degree in 1990. His research interests include optimal control problems for the partial differential equations, variation calculus.