TWMS Jour. Pure Appl. Math., V.3, N.1, 2012, pp.75-91

BOUNDARY VALUE PROBLEMS FOR PSEUDOHYPERBOLIC EQUATIONS WITH A VARIABLE TIME DIRECTION*

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Abstract. In this paper we study the solvability of boundary value problems for pseudo-hyperbolic equations with a discontinuous coefficient at the highest time derivative and forward-backward time $\operatorname{sgn} x u_{tt} - u_{xxt} + c(x,t)u = f(x,t)$. We establish the existence and uniqueness theorems.

Keywords: boundary value problems, pseudohyperbolic equation, time direction change, regular solutions.

AMS Subject Classification: 35M99.

1. Introduction

This paper is an investigation of the solvability of boundary value problems for the one-dimensional linear pseudohyperbolic equation with variable time direction. The equations with variable time direction arise in many applied problems. For example, such equation arises in describing the electrons diffusion process (in a plate), in hydrodynamics (the study of fluid motion with an alternating viscosity) and many other physical processes. The first work devoted to parabolic equations with changing time direction was the work of M. Gevrey [1]. The most intensive exploring of such kind equations began in 1960-70. S.A. Tersenov in several studies investigated a contact parabolic equation with changing time direction

$$\operatorname{sgn} x\, u_t = u_{xx}, \quad (x,t) \in (-1,1) \times (0,T),$$

and a number of other model equations, which are reduced to a system of singular integral equations with the help of potential theory [6]. In these papers necessary and sufficient conditions for solvability in spaces $H_{x\ t}^{p,p/2}$, $p > 2$ were obtained. In this case, the orthogonality conditions have been written explicitly and the number of necessary conditions of orthogonality is finite. But in the multidimensional case, the number of orthogonality conditions (of integral character) is infinite. First time this fact was noted by S.G. Pyatkov [5].

Further, boundary value problems for parabolic equations with changing time direction are considered in works of S.V. Popov and his students [3, 4]. They have discharged necessary and sufficient conditions for solvability in Holder spaces in the same way as S.A. Tersenov. The fulfilment of these conditions will increase the smoothness of solutions if the smoothness of data problem is increased. Moreover, for higher-order equations the smoothness of solution depends on the sewing conditions, namely, one can establish dependence between the index of Holder spaces and the coefficients of sewing conditions.

* The work is executed within the limits of realization the federal target program "Scientific and scientific-pedagogical shots of Innovative Russia" for 2009-2013 on action 1.3.1

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Manuscript received 14 July 2011.

In work of A.I. Kozhanov [2] by the regularization method and the method of continuation parameter the existence of regular solutions of the first boundary value problem for Sobolev's type equations of one class - namely, for pseudohyperbolic equations with variable directions and continuous coefficient is proved. In this paper, using the same method we prove the regular solvability of the boundary value problem for pseudohyperbolic equations with variable time direction and discontinuous coefficients. Note that the regularization method and the method of continuation parameter gives a new approach in researching the boundary value problems for evolution equations with variable direction of time.

2. Problem statement

Let Q be a rectangle $(-1,1)\times(0,T)$, $0<T<+\infty$, $c(x,t)$ and $f(x,t)$, $(x,t)\in\overline{Q}$— given functions, α and β — given real numbers. Denote

$$Q^+=\{(x,t):\ (x,t)\in Q,\ x>0\},\ Q^-=\{(x,t):\ (x,t)\in Q,\ x<0\},\ Q_1=Q^+\cup Q^-.$$

Boundary value problem I: *find a solution $u(x,t)$, $(x,t)\in Q_1$ of equation*

$$sgn\, x\, u_{tt}-u_{xxt}+c(x,t)u=f(x,t) \tag{1}$$

satisfying the boundary conditions

$$u(-1,t)=u(1,t)=0,\quad 0<t<T, \tag{2}$$

$$u(x,0)=u_t(x,0)=0,\quad 0<x<1, \tag{3}$$

$$u(x,T)=u_t(x,T)=0,\quad -1<x<0, \tag{4}$$

and the sewing conditions

$$u(+0,t)=\alpha u(-0,t),\quad 0<t<T, \tag{5}$$

$$\beta u_x(+0,t)=u_x(-0,t),\quad 0<t<T. \tag{6}$$

Boundary value problem II: *find a solution $u(x,t)$, $(x,t)\in Q_1$ of equation* (1) *satisfying the conditions* (2), (5), (6) *and*

$$u(x,T)=u_t(x,0)=0,\quad 0<x<1, \tag{7}$$

$$u(x,0)=u_t(x,T)=0,\quad -1<x<0. \tag{8}$$

Boundary value problem III: *find a solution $u(x,t)$, $(x,t)\in Q_1$ of equation* (1) *satisfying the conditions* (2), (5), (6) *and*

$$u(x,0)=u(x,T)=0,\quad -1<x<1. \tag{9}$$

Let $c_1(x,t)$, $c_2(x,t)$, $f_1(x,t)$ and $f_2(x,t)$ be given functions where $(x,t)\in\overline{Q^+}$. Further, we consider tasks associated with boundary value problems I – III.

Boundary value problem I': *find a solutions $u(x,t)$ and $v(x,t)$, $(x,t)\in Q^+$ of equations*

$$u_{tt}-u_{xxt}+c_1(x,t)u=f_1(x,t) \tag{10}$$

$$-v_{tt}-v_{xxt}+c_2(x,t)v=f_2(x,t) \tag{11}$$

satisfying the boundary conditions

$$u(1,t)=0,\ v(1,t)=0,\quad 0<t<T, \tag{12}$$

$$u(x,0)=u_t(x,0)=0,\qquad 0<x<1, \tag{13}$$

$$v(x,T)=v_t(x,T)=0,\qquad 0<x<1, \tag{14}$$

and the sewing conditions

$$u(0,t) = \alpha v(0,t), \;\; v_x(0,t) = -\beta u_x(0,t), \;\; 0 < t < T. \tag{15}$$

Boundary value problem II': *find a solutions* $u(x,t)$ *and* $v(x,t)$, $(x,t) \in Q^+$ *of equations* (10) *and* (11) *respectively, satisfying the boundary conditions* (12), *the sewing conditions* (15), *conditions* (7) *and*

$$v(x,0) = v_t(x,T) = 0, \quad 0 < x < 1. \tag{16}$$

Boundary value problem III': *find a solutions* $u(x,t)$ *and* $v(x,t)$, $(x,t) \in Q^+$ *of equations* (10) *and* (11) *respectively, satisfying the boundary conditions* (12), *the sewing conditions* (15) *and*

$$u(x,0) = u(x,T) = 0, \quad 0 < x < 1, \tag{17}$$

$$v(x,0) = v(x,T) = 0, \quad 0 < x < 1. \tag{18}$$

Using solutions of boundary value problems $I' - III'$, solutions of boundary value problems I — III will be constructed.

3. Solvability of boundary value problem I'

Let V_0^+ be a linear space

$$V_0^+ = \{v(x,t) : \, v(x,t) \in L_2(Q^+), \, v_{tt}(x,t) \in L_2(Q^+), \, v_{xxt}(x,t) \in L_2(Q^+)\}.$$

Let's put the following norm in this space

$$\|v\|_{V_0^+} = \left(\int\limits_{Q^+} (v^2 + v_{tt}^2 + v_{xxt}^2) \, dx \, dt \right)^{\frac{1}{2}},$$

it is obvious that the space V_0^+ with such norm is a Banach space.

Theorem 3.1. *Let the following conditions*

$$\alpha\beta > 0; \tag{19}$$

$$c_1(x,t) = c_{11}(x,t) + c_{12}(x,t), \quad c_2(x,t) = c_{21}(x,t) + c_{22}(x,t),$$

$$c_{11}(x,t) \in C^1(\overline{Q^+}), \quad c_{21}(x,t) \in C^1(\overline{Q^+}),$$

$$c_{12}(x,t) \in C(\overline{Q^+}), \quad c_{22}(x,t) \in C(\overline{Q^+}),$$

$$c_{11}(x,T) \geq 0, \quad c_{21}(x,0) \leq 0 \quad at \quad x \in [0,1],$$

$$c_{11t}(x,t) \leq 0, \quad c_{21t}(x,t) \leq 0 \quad at \quad (x,t) \in \overline{Q^+},$$

$$T^2 \max_{\overline{Q^+}} c_{12}^2(x,t) < 1, \quad T^2 \max_{\overline{Q^+}} c_{22}^2(x,t) < 1 \tag{20}$$

hold. Then the boundary value problem I' *can not have more than one solution* $(u(x,t), v(x,t))$ *such that* $u(x,t) \in V_0^+$, $v(x,t) \in V_0^+$.

Proof. Let $(u(x,t), v(x,t))$ be a solution of boundary value problem I' such that $u(x,t) \in V_0^+$, $v(x,t) \in V_0^+$. We multiply the equation (10) by a function $u_t(x,t)$, the equation (11) by $\gamma v_t(x,t)$, where $\gamma = \frac{\alpha}{\beta}$, then we integrate the received equalities over the rectangle Q^+ and add the results. Using the representation of functions $c_1(x,t)$, $c_2(x,t)$ and integrating by parts, we obtain the following equality

$$\int\limits_{Q^+} [u_{xt}^2 + \gamma v_{xt}^2 - \frac{1}{2} c_{11t} u^2 - \frac{\gamma}{2} c_{21t} v^2]\, dx\, dt +$$

$$+\frac{1}{2}\int\limits_0^1 [u_t^2(x,T) + \gamma v_t^2(x,0) + c_{11}(x,T)u^2(x,T) - \gamma c_{21}(x,0)v^2(x,0)]\, dx +$$

$$+\int\limits_0^T u_{xt}(0,t)u_t(0,t)\, dt + \gamma \int\limits_0^T v_{xt}(0,t)v_t(0,t)\, dt =$$

$$= \int\limits_{Q^+} [f_1 u_t + \gamma f_2 v_t - c_{12} u u_t - \gamma c_{22} v v_t]\, dx\, dt.$$

Note that, by (19), the number γ is positive. Further, the sewing conditions (15) give the equality

$$\int\limits_0^T u_{xt}(0,t)u_t(0,t)\, dt + \gamma \int\limits_0^T v_{xt}(0,t)v_t(0,t)\, dt = 0.$$

Taking into account condition (20), elementary inequalities

$$\int\limits_{Q^+} u^2\, dx\, dt \le T^2 \int\limits_{Q^+} u_t^2\, dx\, dt, \qquad \int\limits_{Q^+} u_t^2\, dx\, dt \le T^2 \int\limits_{Q^+} u_{xt}^2\, dx\, dt,$$

and Young's inequality, we easily obtain the next estimate

$$\int\limits_{Q^+} [u_{xt}^2 + v_{xt}^2]\, dx\, dt + \int\limits_0^1 [u_t^2(x,T) + v_t^2(x,0)]\, dx \le C_0 \int\limits_{Q^+} [f_1^2 + f_2^2]\, dx\, dt \tag{21}$$

with constant C_0, defined by numbers α, β, T and functions $c_{12}(x,t)$, $c_{22}(x,t)$. According to this estimate and the conditions (12) — (14), functions $u(x,t)$ and $v(x,t)$ are identically equal to zero at Q^+ if the $f_1(x,t) \equiv f_2(x,t) \equiv 0$. Thus it means that the solution $(u(x,t), v(x,t))$ of boundary value problem I' is unique.

□

Now we investigate the solvability of boundary value problem I'.

Theorem 3.2. *Let the conditions* (19) *and* (20) *hold. Let, besides, the following conditions*

$$c_i(x,t) \in C^1(\overline{Q^+}), \quad c_i(0,t) = 0 \quad at \quad t \in [0,T]; \quad f_i(x,t) \in L_2(Q^+),$$

$$f_{ix}(x,t) \in L_2(Q^+), \quad f_i(0,t) = f_i(1,t) = 0 \quad at \quad t \in [0,T], \quad i = 1,2 \tag{22}$$

hold. Then the boundary value problem I' has solution $(u(x,t), v(x,t))$ such that $u(x,t) \in V_0^+$, $v(x,t) \in V_0^+$.

Proof. We use the regularization method and the method of parameter continuation.

Let ε_0 be positive number which value will be specified below, ε is a number from the interval $(0, \varepsilon_0)$. Now, we consider the boundary value problem: *find a solutions $u(x,t)$ and $v(x,t)$, $(x,t) \in Q_1$ of equations*

$$-\varepsilon u_{xxtt} + u_{tt} - u_{xxt} + c_1(x,t)u = f_1(x,t), \tag{10_ε}$$

$$\varepsilon v_{xxtt} - v_{tt} - v_{xxt} + c_2(x,t)v = f_2(x,t) \tag{11_ε}$$

satisfying the conditions (12) — (15). Let's establish its solvability.

Let V_1 be a linear space

$$V_1 = \{v(x,t): \quad v(x,t) \in V_0^+, \quad v_{xtt}(x,t) \in L_2(Q^+), \quad v_{xxtt}(x,t) \in L_2(Q^+)\},$$

with norm

$$\|v\|_{V_1} = \|v\|_{V_0^+} + \|v_{xtt}\|_{L_2(Q^+)} + \|v_{xxtt}\|_{L_2(Q^+)}.$$

We show that the boundary value problem (10_ε), (11_ε), (12) — (15) is solvable in the space V_1 for any functions $f_1(x,t)$, $f_2(x,t)$ such that $f_1(x,t) \in L_2(Q^+)$, $f_2(x,t) \in L_2(Q^+)$ when ε is fixed. We use the method of parameter continuation.

Let λ be a number from interval [0,1]. We consider the family of boundary value problems: *find a solutions $u(x,t)$ and $v(x,t)$, $(x,t) \in Q^+$ of equations* (10_ε) *and* (11_ε) *respectively, satisfying the conditions* (12) — (14) *and*

$$u(0,t) = \lambda\alpha v(0,t), \quad v_x(0,t) = -\lambda\beta u_x(0,t), \quad 0 < t < T. \tag{15_λ}$$

Denote by Λ the set of integers λ in the interval [0,1] for which the boundary value problem (10_ε), (11_ε), (12) — (14), (15_λ) has solution from V_1 when ε is fixed. If it turns out that the set Λ is not empty, open and closed, then it will coincide with the whole interval [0,1] (see [7]).

Boundary value problem (10_ε), (11_ε), (12) — (14), (15_0) is solvable in space V_1 (see [8]). It follows that the number 0 belongs to Λ and thus the set Λ is not empty.

To proof the the openness and closure of the set Λ it is enough to show for fixed ε that there is a uniform, with respect to λ, a priori estimate

$$\|u\|_{V_1} + \|v\|_{V_1} \leq N(\|f_1\|_{L_2(Q^+)} + \|f_2\|_{L_2(Q^+)}) \tag{23}$$

for all solutions $u(x,t)$, $v(x,t)$ of boundary value problem (10_ε), (11_ε), (12) — (14), (15_λ) such that $u(x,t) \in V_1$, $v(x,t) \in V_1$.

Let's show that the required estimate actually takes place.

We multiply the equation (10_ε) by a function $u_t(x,t)$, equation (11_ε) by $\gamma v_t(x,t)$, where $\gamma = \frac{\alpha}{\beta}$. Let's integrate the received equalities over the rectangle and add the results. Repeating the calculations by which we obtained the estimate (21), we find that for solutions $u(x,t)$, $v(x,t)$ of boundary value problem (10_ε), (11_ε), (12) — (14), (15_λ) the inequality

$$\int\limits_Q (u_t^2 + v_t^2 + u_{xt}^2 + v_{xt}^2)\,dx\,dt \leq N_1 \left\{ \varepsilon^2 \int\limits_{Q^+} [u_{xxtt}^2 + v_{xxtt}^2]\,dx\,dt + \int\limits_{Q^+} [f_1^2 + f_2^2]\,dx\,dt \right\}, \tag{24}$$

is carried out, where the number N_1 is defined by numbers α, β, T and functions $c_{12}(x,t)$, $c_{22}(x,t)$.

Further, we multiply the equation (10_ε) by a function $-u_{xxtt}$, equation (11_ε) — γv_{xxtt}, $\gamma = \frac{\alpha}{\beta}$, and integrate the received equalities over the rectangle Q^+ and add the results. Integrating by parts and using the conditions (12)–(14), (15_λ) and $c_1(0,t) = c_2(0,t) = 0$ we obtain the following equality

$$\int\limits_{Q^+} [\varepsilon u_{xxtt}^2 + \gamma\varepsilon v_{xxtt}^2 + u_{xtt}^2 + \gamma v_{xtt}^2]\,dx\,dt + \frac{1}{2}\int\limits_0^1 u_{xxt}^2(x,T)\,dx + \frac{\gamma}{2}\int\limits_0^1 v_{xxt}^2(x,0)\,dx =$$

$$= -\int\limits_{Q^+} f_1 u_{xxtt}\,dx\,dt + \gamma\int\limits_{Q^+} f_2 v_{xxtt}\,dx\,dt - \int\limits_{Q^+} c_1 u_x u_{xtt}\,dx\,dt - \int\limits_{Q^+} c_{1x} u u_{xtt}\,dx\,dt +$$

$$+\gamma\int\limits_{Q^+} c_2 v_x v_{xtt}\,dx\,dt + \gamma\int\limits_{Q^+} c_{2x} v v_{xtt}\,dx\,dt. \tag{25}$$

We estimate the first two terms on the right side of (25) with the help of the Young's inequality

$$\left| -\int\limits_{Q^+} f_1 u_{xxtt}\,dx\,dt + \gamma\int\limits_{Q^+} f_2 v_{xxtt}\,dx\,dt \right| \le \frac{\varepsilon}{2}\int\limits_{Q^+} u_{xxtt}^2\,dx\,dt + \frac{\gamma\varepsilon}{2}\int\limits_{Q^+} v_{xxtt}^2\,dx\,dt +$$

$$+\frac{1}{2\varepsilon}\int\limits_{Q^+} f_1^2\,dx\,dt + \frac{\gamma}{2\varepsilon}\int\limits_{Q^+} f_2^2\,dx\,dt. \tag{26}$$

We estimate the remaining terms on the right side of (25) again using the Young's inequality, then the received integrals – with the help of the elementary integral inequalities given above, and the inequality (24). We obtain the following estimate

$$\Bigg| -\int\limits_{Q^+} c_1 u_x u_{xtt}\,dx\,dt - \int\limits_{Q^+} c_{1x} u u_{xtt}\,dx\,dt + \gamma\int\limits_{Q^+} c_2 v_x v_{xtt}\,dx\,dt +$$

$$+\gamma\int\limits_{Q^+} c_{2x} v v_{xtt}\,dx\,dt \Bigg| \le \frac{1}{2}\int\limits_{Q^+} u_{xtt}^2\,dx\,dt + \frac{\gamma}{2}\int\limits_{Q^+} v_{xtt}^2\,dx\,dt +$$

$$+N_2\left\{ \varepsilon^2\int\limits_{Q^+} [u_{xxtt}^2 + v_{xxtt}^2]\,dx\,dt + \int\limits_{Q^+} [f_1^2 + f_2^2]\,dx\,dt \right\}, \tag{27}$$

where constant N_2 is defined by numbers α, β, T and functions $c_1(x,t)$, $c_2(x,t)$.

Let the number ε_0 be such that the inequality $2N_2\varepsilon < 1$ holds for $\varepsilon < \varepsilon_0$. Then a consequence of equality (25) and inequalities (26) and (27) will be an a priory estimate of solutions of the boundary value problem (10_ε), (11_ε), (12) — (14), (15_λ):

$$\varepsilon\int\limits_{Q^+} [u_{xxtt}^2 + v_{xxtt}^2]\,dx\,dt + \int\limits_{Q^+} [u_{xtt}^2 + v_{xtt}^2]\,dx\,dt \le N_3\int\limits_{Q^+} [f_1^2 + f_2^2]\,dx\,dt, \tag{28}$$

where constant N_3 is defined by numbers α, β, T, ε and functions $c_1(x,t)$, $c_2(x,t)$.

According to the estimate (28) and the inequality (24) the required estimate (23) is obviously true.

Let's show that the given estimate is followed by the openness and closure of set Λ.

Define the functions $w(x,t)$ and $z(x,t)$:

$$w(x,t) = u(x,t) - \lambda\alpha(1-x)v(0,t), \quad z(x,t) = v(x,t) + \lambda\beta(x-1)u_x(0,t).$$

The functions $u(x,t)$ and $v(x,t)$ are uniquely computed through the functions $w(x,t)$ and $z(x,t)$:

$$u(x,t) = w(x,t) + \frac{\lambda^2\alpha\beta(1-x)}{1+\lambda^2\alpha\beta} w_x(0,t) + \frac{\lambda\alpha(1-x)}{1+\lambda^2\alpha\beta} z(0,t),$$

$$v(x,t) = z(x,t) - \frac{\lambda\beta(x-1)}{1+\lambda^2\alpha\beta} w_x(0,t) + \frac{\lambda^2\alpha\beta(x-1)}{1+\lambda^2\alpha\beta} z(0,t);$$

besides, it is obvious that next equalities

$$w(0,t) = 0, \quad z_x(0,t) = 0, \quad t \in (0,T) \tag{29}$$

are carried out. Further, the equations (10_ε) and (11_ε) will be transformed to the following equations for functions $w(x,t)$ and $z(x,t)$:

$$-\varepsilon w_{xxtt} + w_{tt} - w_{xxt} + c_1(x,t)w = f_1(x,t) - \frac{\lambda^2\alpha\beta(1-x)}{1+\lambda^2\alpha\beta} w_{xtt}(0,t) -$$

$$- \frac{\lambda\alpha(1-x)}{1+\lambda^2\alpha\beta} z_{tt}(0,t) - \frac{\lambda^2\alpha\beta(1-x)c_1(x,t)}{1+\lambda^2\alpha\beta} w_x(0,t) - \frac{\lambda\alpha(1-x)c_1(x,t)}{1+\lambda^2\alpha\beta} z(0,t), \tag{$10'_\varepsilon$}$$

$$\varepsilon z_{xxtt} - z_{tt} - z_{xxt} + c_2(x,t)z = f_2(x,t) + \frac{\lambda\beta(x-1)}{1+\lambda^2\alpha\beta} w_{xtt}(0,t) -$$

$$- \frac{\lambda^2\alpha\beta(x-1)}{1+\lambda^2\alpha\beta} z_{tt}(0,t) + \frac{\lambda\beta(x-1)c_2(x,t)}{1+\lambda^2\alpha\beta} w_x(0,t) - \frac{\lambda^2\alpha\beta(x-1)c_2(x,t)}{1+\lambda^2\alpha\beta} z(0,t). \tag{$11'_\varepsilon$}$$

Equations ($10'_\varepsilon$), ($11'_\varepsilon$) together with conditions (29), (12)–(14) give a boundary value problem for functions $w(x,t)$ and $z(x,t)$, which is equivalent to a problem (10_ε), (11_ε), (12)- (14), (15_λ) because of one-to-one correspondence between functions $u(x,t)$, $v(x,t)$ and $w(x,t)$, $z(x,t)$. For solutions $w(x,t)$, $z(x,t)$ of this problem the estimate (23) will remain. According to this estimate and continuity with respect to parameter λ of the family problems ($10'_\varepsilon$), ($11'_\varepsilon$), (12)– (14), (29) the openness and closure of set Λ follows (see [7]). It is worthy of note that at first we establish the openness and closure of set Λ for family of problems ($10'_\varepsilon$), ($11'_\varepsilon$), (12)–(14), (29) and then the openness and closure of set Λ for problems (10_ε), (11_ε), (12)–(14), (15_λ)) follows by itself.

So, at fixed ε from the interval $(0, \varepsilon_0)$, the set Λ isn't empty, open and closed, and, thereby, coincides with whole interval [0,1]. Hence, a boundary value problem (10_ε), (11_ε), (12)–(15) at fixed ε is solvable in space V_1 for any functions $f_1(x,t)$ and $f_2(x,t)$ such that $f_1(x,t) \in L_2(Q^+)$, $f_2(x,t) \in L_2(Q^+)$. We will show that for the family of solutions $\{u_\varepsilon(x,t), v_\varepsilon(x,t)\}$ of this problem an a priory estimate takes place. This estimate is both uniform with respect to parameter ε and with its help it will be possible to organize the passage to the limit.

First of all we will notice that for the family $\{u_\varepsilon(x,t), v_\varepsilon(x,t)\}$ inequality (24) takes place. Further, for the first two terms on the right side of equality (25) we will execute integration by parts with respect to variable x.

Applying an Young's inequality, using condition (22) and inequality (27), we receive a following inequality

$$\varepsilon \int\limits_{Q^+} [u^2_{\varepsilon xxtt} + v^2_{\varepsilon xxtt}]\, dx\, dt + \int\limits_{Q^+} [u^2_{\varepsilon xtt} + v^2_{\varepsilon xtt}]\, dx\, dt \leq$$

$$\leq N_4 \left\{ \varepsilon^2 \int\limits_{Q^+} [u^2_{\varepsilon xxtt} + v^2_{\varepsilon xxtt}]\, dx\, dt + \int\limits_{Q^+} [f_1^2 + f_{1x}^2 + f_2^2 + f_{2x}^2]\, dx\, dt \right\}, \tag{30}$$

where number N_4 is defined by numbers α, β, T and functions $c_1(x,t)$, $c_2(x,t)$. Reducing number ε_0 so, that inequalities $2N_2\varepsilon < 1$, $2N_4\varepsilon < 1$ hold together for $\varepsilon < \varepsilon_0$, and according to inequalities (24) and (30), we will receive an a priory estimate

$$\varepsilon \int\limits_{Q^+} [u^2_{\varepsilon xxtt} + v^2_{\varepsilon xxtt}]\, dx\, dt + \int\limits_{Q^+} [u^2_{\varepsilon xtt} + v^2_{\varepsilon xtt}]\, dx\, dt + \|u_\varepsilon\|^2_{V_0^+} + \|v_\varepsilon\|^2_{V_0^+} \le$$

$$\le N_0 \left(\|f_1\|^2_{L_2(Q^+)} + \|f_2\|^2_{L_2(Q^+)} + \|f_{1x}\|^2_{L_2(Q^+)} + \|f_{2x}\|^2_{L_2(Q^+)} \right), \tag{31}$$

where the constant N_0 is defined by numbers α, β, T and functions $c_1(x,t)$, $c_2(x,t)$.

According to the estimate (31) and properties of reflexivity of the space L_2 it follows that there exist such sequence $\{\varepsilon_n\}$ and functions $u(x,t)$, $v(x,t)$ that $\varepsilon_n \to 0$, $u_{\varepsilon_n}(x,t) \to u(x,t)$, $v_{\varepsilon_n}(x,t) \to v(x,t)$ weakly in space $W_2^2(Q^+)$, $u_{\varepsilon_n xxt}(x,t) \to u_{xxt}(x,t)$, $v_{\varepsilon_n xxt}(x,t) \to v_{xxt}(x,t)$, $\varepsilon_n u_{\varepsilon_n xxtt}(x,t) \to 0$, $\varepsilon_n v_{\varepsilon_n xxtt}(x,t) \to 0$ weakly in space $L_2(Q^+)$, $u_{\varepsilon_n}(0,t) \to u(0,t)$, $v_{\varepsilon_n}(0,t) \to v(0,t)$ weakly in space $W_2^2([0,T])$, $u_{\varepsilon_n x}(0,t) \to u_x(0,t)$, $v_{\varepsilon_n x}(0,t) \to v_x(0,t)$ weakly in space $W_2^1([0,T])$ when $n \to \infty$. It is obvious that limiting functions $u(x,t)$ and $v(x,t)$ will belong to space V_0^+, equations (10) and (11), boundary conditions (12)–(15) will be carried out for them. In other words, functions $u(x,t)$ and $v(x,t)$ will give the solution of a boundary value problem I' from a required class. □

4. Solvability of boundary value problem II'

Let us first discuss the uniqueness of solutions.

Theorem 4.1. *Let the following conditions*

$$\alpha\beta > 0, \tag{32}$$

$$c_1(x,t) \in C^1(\overline{Q^+}), \quad c_2(x,t) \in C^1(\overline{Q^+}), \quad c_1(x,t) \le 0, \quad c_2(x,t) \ge 0,$$

$$T \max_{\overline{Q^+}} |c_{1x}(x,t)| < 1, \quad T \max_{\overline{Q^+}} |c_{2x}(x,t)| < 1, \tag{33}$$

$$c_1(0,t) + c_2(0,t) = 0 \quad at \quad t \in [0,T] \tag{34}$$

hold. Then the boundary value problem II' cannot have more than one solution $\{u(x,t), v(x,t)\}$ such that $u(x,t) \in V_0^+$, $v(x,t) \in V_0^+$.

Proof. Let $f_1(x,t) \equiv 0$, $f_2(x,t) \equiv 0$, and let $\{u(x,t), v(x,t)\}$ be solutions of the boundary value problem II' with the functions $f_1(x,t)$, $f_2(x,t)$.

We multiply the equation (10) by function $u_{xx}(x,t)$, equation (11) by $\gamma v_{xx}(x,t)$, $\gamma = \frac{\alpha}{\beta}$ and integrate the received equalities over the rectangle Q^+ then sum up the results. Integrating by parts and using the conditions (7), (12), (16), (15) and also (32), (33), we obtain the next inequality

$$\int\limits_{Q^+} [u^2_{xt} + v^2_{xt}]\, dx\, dt + \int\limits_0^1 [u^2_{xx}(x,0) + v^2_{xx}(x,T)]\, dx \le 0.$$

This inequality gives $u(x,t) \equiv 0$, $v(x,t) \equiv 0$ when $(x,t) \in Q^+$.

□

Theorem 4.2. *Let the following conditions*

$$\alpha\beta < 0, \tag{35}$$

$$c_1(x,t) = c_{11}(x,t) + c_{12}(x,t), \quad c_2(x,t) = c_{21}(x,t) + c_{22}(x,t),$$

$$c_{ij}(x,t) \in C(\overline{Q^+}), \quad i,j = 1,2, \quad c_{11}(x,t) \le 0, \quad c_{21}(x,t) \ge 0,$$

$$T^2 \max_{\overline{Q^+}} |c_{12}(x,t)| < 1, \quad T^2 \max_{\overline{Q^+}} |c_{22}(x,t)| < 1 \tag{36}$$

hold. Then the boundary value problem II' can not have more than one solution $\{u(x,t), v(x,t)\}$ such that $u(x,t) \in V_0^+$, $v(x,t) \in V_0^+$.

Proof. We multiply the equation (10) by a function $-u(x,t)$, equation (11) — $\gamma v(x,t)$, $\gamma = -\frac{\alpha}{\beta}$ and integrate the received equalities over the rectangle Q^+ than sum up the results. Let $f_1(x,t) \equiv 0$, $f_2(x,t) \equiv 0$ then integrating by parts and using the conditions (7), (12), (16), (15) and also (35), (36) we obtain the next inequality

$$\int\limits_{Q^+} [u_t^2 + v_t^2]\, dx\, dt + \int\limits_0^1 [u_x^2(x,0) + v_x^2(x,T)]\, dx \le 0.$$

From this inequality the uniqueness is carried out.

□

Let's consider the solvability of boundary value problem II'.

Theorem 4.3. *Let the following conditions* (32) *and* (33) *hold, and the following conditions*

$$c_i(0,t) = 0 \quad at \quad t \in [0,T] \quad i = 1,2; \tag{37}$$

$$f_i(x,t) \in L_2(Q^+), \quad f_{ix}(x,t) \in L_2(Q^+), \quad i = 1,2;$$
$$f_i(0,t) = f_i(1,t) = 0 \quad at \quad t \in [0,T], \quad i = 1,2 \tag{38}$$

also hold. Then the boundary value problem II' has solution $(u(x,t), v(x,t))$ such that $u(x,t) \in V_0^+$, $v(x,t) \in V_0^+$.

Proof. Let's use the regularization method and the method of parameter continuation again.

Let ε be a positive number. Now, we consider the boundary value problem: *find a solutions $u(x,t)$ and $v(x,t)$, $(x,t) \in Q^+$ of equations* (10_ε) *and* (11_ε) *respectively, satisfying the conditions (7), (12), (15), (16).* Let ε be fixed. Then let's show that the boundary value problem (10_ε), (11_ε), (7), (12), (15), (16) is solvable in space V_1 for any functions $f_1(x,t)$, $f_2(x,t)$ such that $f_1(x,t) \in L_2(Q^+)$, $f_2(x,t) \in L_2(Q^+)$. We use the method of parameter continuation again.

Let λ be a number from the interval $[0,1]$. Let's investigate the family of boundary value problems: *find a solutions $u(x,t)$ and $v(x,t)$, $(x,t) \in Q^+$ of equations* (10_ε) *and* (11_ε) *respectively, satisfying conditions* (7), (12), (15_λ) *and* (16). Note that, the boundary value problem (10_ε), (11_ε), (7), (12), (15_0), (16) has a solution in V_1 for any functions $f_1(x,t)$, $f_2(x,t)$ such that $f_1(x,t) \in L_2(Q^+)$ and $f_2(x,t) \in L_2(Q^+)$ when ε is fixed. Therefore, to establish the solvability in space V_1 of all boundary value problems (7), (12), (15_λ), (16) it is enough to prove the estimate (23).

We multiply the equation (10_ε) by a function $u_{xx}(x,t)$, equation (11_ε) — $-\gamma v_{xx}(x,t)$ where $\gamma = \frac{\alpha}{\beta}$ and integrate the received equalities over the rectangle Q^+ and sum up the results. Integrating by parts, using conditions (7), (12), (15_λ), (16), (32) — (34) and Young's inequality, we obtain the following inequality

$$\varepsilon \int\limits_{Q^+} [u_{xxt}^2 + v_{xxt}^2]\, dx\, dt + \int\limits_{Q^+} [u_{xt}^2 + v_{xt}^2]\, dx\, dt + \int\limits_0^1 [u_{xx}^2(x,0) + v_{xx}^2(x,T)]\, dx \le$$

$$\le \delta_1 \int\limits_{Q^+} [u_{xx}^2 + v_{xx}^2]\, dx\, dt + C(\delta_1) \int\limits_{Q^+} [f_1^2 + f_2^2]\, dx\, dt, \tag{39}$$

where δ_1 is an arbitrary positive number, and the number $C(\delta_1)$ is defined by functions $c_1(x,t)$, $c_2(x,t)$ and number T.

Further, we multiply equation (10_ε) by a function $(-u_{xxtt})$, equation (11_ε) — by a function γv_{xxtt}, $\gamma = \frac{\alpha}{\beta}$ and integrate the received equalities over the rectangle Q^+ and sum up the results. Integrating by parts and using the conditions (7), (12), (15_λ), (16), (32) — (34), elementary integral inequalities, the Young's inequality and the estimate (39), we obtain the following inequality

$$\int\limits_{Q^+} [u_{xxtt}^2 + v_{xxtt}^2 + u_{xtt}^2 + v_{xtt}^2]\,dx\,dt \le \delta_2 \int\limits_{Q^+} [u_{xx}^2 + v_{xx}^2]\,dx\,dt + K(\delta_2) \int\limits_{Q^+} [f_1^2 + f_2^2]\,dx\,dt, \qquad (40)$$

where δ_2 is an arbitrary positive number, and the number $K(\delta_2)$ is defined by functions $c_1(x,t)$, $c_2(x,t)$, and numbers T, ε.

From inequalities (39), (40) with the help of elementary integral inequalities and choosing number δ_2 as a small it is not difficult to deduce the desired estimate (23).

So, for the solutions of boundary value problems (10_ε), (11_ε), (7), (12), (15_λ), (16) there is a uniform with respect to λ estimate (23). According to this estimate and the solvability in space V_1 of boundary value problem (10_ε), (11_ε), (12), (15_0), (16), the possibility of applying the theorem about the method of parameter continuation [7] follows (for more details see the proof of theorem 2).

Let $\{u(x,t), v(x,t)\}$ be a solution of boundary value problem (10_ε), (11_ε), (7), (12), (15), (16).

Now, we multiply the equation (10_ε) by a function $u_{xx}(x,t)$ and the equation (11_ε) by $(-\gamma v_{xx}(x,t))$. Integrating by parts and using conditions (7), (12), (15), (16), (32) — (34), (37), (38), applying the Young's inequality we received the next estimate

$$\varepsilon \int\limits_{Q^+} [u_{xxt}^2 + v_{xxt}^2]\,dx\,dt + \int\limits_{Q^+} [u_{xt}^2 + v_{xt}^2]\,dx\,dt \le M_1 \int\limits_{Q^+} [f_{1x}^2 + f_{2x}^2]\,dx\,dt, \qquad (41)$$

where the number M_1 is defined by functions $c_1(x,t)$ and $c_2(x,t)$, and constant T.

Further, we multiply equation (10_ε) by a function $-u_{xxtt}$, equation (11_ε) by γv_{xxtt}, where $\gamma = \frac{\alpha}{\beta}$ and integrate the received equalities over the rectangle Q^+ and sum up the results. Integrating by parts and using the conditions (7), (12), (15_λ), (16), (37), (38), the Young's inequality and the estimate (41) we obtain a uniform with respect to ε estimate

$$\varepsilon \int\limits_{Q^+} [u_{xxtt}^2 + v_{xxtt}^2]\,dx\,dt + \int\limits_{Q^+} [u_{xtt}^2 + v_{xtt}^2]\,dx\,dt \le M_2 \int\limits_{Q^+} [f_{1x}^2 + f_{2x}^2]\,dx\,dt, \qquad (42)$$

where the constant M_2 is defined by functions $c_1(x,t)$, $c_2(x,t)$, and number T.

According to estimates (41), (42) and equations (10_ε), (11_ε), the third uniform, with respect to ε, estimate

$$\int\limits_{Q^+} [u_{xxt}^2 + v_{xxt}^2]\,dx\,dt \le M_3 \int\limits_{Q^+} [f_1^2 + f_2^2 + f_{1x}^2 + f_{2x}^2]\,dx\,dt, \qquad (43)$$

holds, where constant M_3 is defined by functions $c_1(x,t)$, $c_2(x,t)$, and number T.

To organize the passage to the limit it is enough to have estimates (41) — (43) (see the proof of theorem 2), limiting functions $u(x,t)$ and $v(x,t)$ will give a solution of boundary value problem II'.

$\square$

Let W_0^+, W_1^+ be a linear spaces

$$W_0^+ = \{w(x,t) : w_{xxtt} \in L_2(Q^+),\ w_{xxxt}(x,t) \in L_2(Q^+)\},$$

$$W_1 = \{w(x,t) : w(x,t) \in W_0^+,\ w_{xxxxtt}(x,t) \in L_2(Q^+).\}$$

Let these spaces be equipped with the norm

$$\|w\|_{W_0^+} = \|w_{xxtt}\|_{L_2(Q^+)} + \|w_{xxxt}\|_{L_2(Q^+)},$$

$$\|w\|_{W_1} = \|w\|_{W_0^+} + \|w_{xxxxtt}\|_{L_2(Q^+)}.$$

Theorem 4.4. *Let the conditions (35) and*

$$c_1(x,t) \le 0, \quad c_2(x,t) \ge 0, \quad c_{1t}(x,0) \ge 0, \quad c_{2t}(x,T) \ge 0, \quad c_{1tt}(x,t) \ge 0,$$

$$c_{2tt}(x,t) \le 0, \quad T^4 \max_Q c_{1x}^2(x,t) < 1, \quad T^4 \max_Q c_{2x}^2(x,t) < 1; \tag{44}$$

$$c_i(0,t) = c_{ix}(0,t) = 0,\ f_i(0,t) = f_{ix}(0,t) = f_i(1,t) = 0,\ \text{at } t \in [0,T]; \tag{45}$$

$$f_i(x,t) \in L_2(Q^+), \quad f_{ixx}(x,t) \in L_2(Q^+),\ i = 1,2 \tag{46}$$

hold. Then the boundary value problem II' has a solution $(u(x,t), v(x,t))$ such that $u(x,t) \in W_0^+$, $v(x,t) \in W_0^+$.

Proof. We use the regularization method and the method of parameter continuation again.

Let ε_0 be a positive number which value will be specified below, ε is a number from the interval $(0, \varepsilon_0)$. We consider the boundary value problem: *find a solutions $u(x,t)$, $v(x,t)$, $(x,t) \in Q^+$ of equations*

$$\varepsilon u_{xxxxtt} + u_{tt} - u_{xxt} + c_1 u = f_1, \tag{$10''_\varepsilon$}$$

$$-\varepsilon v_{xxxxtt} - v_{tt} - v_{xxt} + c_2 v = f_2, \tag{$11''_\varepsilon$}$$

satisfying conditions (7), (12), (15), (16) and

$$u_{xx}(1,t) = v_{xx}(1,t) = 0, \quad 0 < t < T, \tag{47}$$

$$u_{xx}(0,t) = -\alpha v_{xx}(0,t),\ v_{xxx}(0,t) = \beta u_{xxx}(0,t). \tag{48}$$

Let's establish its solvability.

We show that the boundary value problem ($10''_\varepsilon$), ($11''_\varepsilon$), (7), (11), (14), (15), (47), (48) is solvable in the space W_1 for any functions $f_1(x,t)$, $f_2(x,t)$ such that $f_1(x,t) \in L_2(Q^+)$, $f_2(x,t) \in L_2(Q^+)$ when ε is fixed.

Let λ be a positive number from $[0,1]$. We consider the family of boundary value problems: *find a solutions $u(x,t)$, $v(x,t)$, $(x,t) \in Q^+$ of equations* ($10''_\varepsilon$), ($11''_\varepsilon$) *respectively, satisfying conditions* (7), (12), (16), (47), (15_λ) *and*

$$u_{xx}(0,t) = -\lambda\alpha v_{xx}(0,t),\ v_{xxx}(0,t) = \lambda\beta u_{xxx}(0,t),\ \ 0 < t < T. \tag{48_λ}$$

The boundary value problem ($10''_\varepsilon$), ($11''_\varepsilon$), (7), (12), (16), (47), (15_0), (48_0) has solution from W_1, when functions $f_1(x,t)$, $f_2(x,t)$ belong to $L_2(Q^+)$ and ε is fixed. Let's show there is a uniform with respect to λ estimate

$$\|u\|_{W_1} + \|v\|_{W_1} \le N(\|f_1\|_{L_2(Q^+)} + \|f_2\|_{L_2(Q^+)}). \tag{49}$$

We multiply the equation ($10''_\varepsilon$) by a function $u_t(x,t)$, equation ($11''_\varepsilon$) by $-\gamma v_t(x,t)$, where $\gamma = -\frac{\alpha}{\beta}$. Let's integrate the received equalities over the rectangle Q^+ and sum up the results. Repeating the calculations by which we obtained the estimate (21), we find out that for solutions

$u(x,t)$, $v(x,t)$ of boundary value problem $(10''_\varepsilon)$, $(11''_\varepsilon)$, (7), (12), (16), (47), (15_λ), (48_λ) the inequality

$$\int\limits_{Q^+} (u_t^2 + v_t^2)\, dx\, dt + \int\limits_0^1 \left(u_x^2(x,0) + v_x^2(x,T)\right)\, dx \leq$$

$$\leq N_1 \left\{ \varepsilon^2 \int\limits_{Q^+} [u_{xxxxtt}^2 + v_{xxxxtt}^2]\, dx\, dt + \int\limits_{Q^+} [f_1^2 + f_2^2]\, dx\, dt \right\}, \tag{50}$$

is carried out, where the number N_1 is defined by numbers α, β, T.

Further, we multiply equation $(10''_\varepsilon)$ by a function u_{xxxxtt}, equation $(11''_\varepsilon)$ by $-\gamma v_{xxxxtt}$, where $\gamma = -\frac{\alpha}{\beta}$.

Let the ε_0 be such that the inequality $2N_2\varepsilon < 1$ is carried out when $\varepsilon < \varepsilon_0$. The number N_2 is defined by numbers α, β, T and functions $c_1(x,t)$, $c_2(x,t)$.

We integrate the received equalities over the rectangle Q^+ and sum up the results. Integrating by parts and using the conditions (7), (12),(16), (47), (15_λ), (48_λ), (44) we obtain the following inequality

$$\varepsilon \int\limits_{Q^+} \left(u_{xxxxtt}^2 + v_{xxxxtt}^2\right)\, dx\, dt + \int\limits_{Q^+} \left(u_{xxtt}^2 + v_{xxtt}^2\right)\, dx\, dt +$$

$$+ \int\limits_0^1 \left[u_{xxxt}^2(x,T) + v_{xxxt}^2(x,0)\right]\, dx\, dt \leq N_3 \int\limits_{Q^+} \left(f_1^2 + f_2^2\right)\, dx\, dt, \tag{51}$$

where the number N_3 is defined by functions $c_1(x,t)$, $c_2(x,t)$ and numbers α, β, T, ε.

According to inequalities (50), (51) and using an elementary inequalities it is easy to deduce the required estimate (49).

So, for a solution of the boundary value problem $(10''_\varepsilon)$, $(11''_\varepsilon)$, (7), (12), (16), (47), (15_λ), (47_λ) there is a uniform with respect to λ estimate (49). From this estimate and solvability in the space W_1 of boundary value problem $(10''_\varepsilon)$, $(11''_\varepsilon)$, (7), (12), (16), (47), (15_0), (48_0) the possibility of applying the theorem about the method of parameter continuation [7] follows (see the proof of theorem 2).

Let $\{u(x,t), v(x,t)\}$ be solution of boundary value problem $(10'')$, $(11''_\varepsilon)$, (7), (12), (15), (16), (47), (48).

We multiply the equation $(10''_\varepsilon)$ by a function $-u_{xxxxtt}(x,t)$, equation $(11''_\varepsilon)$ by γv_{xxxxtt}, where $\gamma = -\frac{\alpha}{\beta}$. Integrating by parts and using the conditions (7), (12), (15), (16), (47), (48), (35), (44)—(46), applying the Young's inequality and using the estimate (50) we received the second uniform, with respect to ε, estimate

$$M(\varepsilon) \int\limits_{Q^+} [u_{xxxxtt}^2 + v_{xxxxtt}^2]\, dx\, dt +$$

$$+ \int\limits_{Q^+} [u_{xxtt}^2 + v_{xxtt}^2]\, dx\, dt \leq M_1 \int\limits_{Q^+} [f_{1xx}^2 + f_{2xx}^2]\, dx\, dt, \tag{52}$$

where the number $M(\varepsilon)$ is infinitesimal when $\varepsilon \to 0$, and the number M_1 is defined by functions $c_1(x,t)$, $c_2(x,t)$ and constant T.

According to estimates (50), (52) and equations ($10''_\varepsilon$), ($11''_\varepsilon$) the third uniform, with respect to ε, estimate

$$\int\limits_{Q^+} [u_t^2 + v_t^2]\, dx\, dt \leq M_3 \int\limits_{Q^+} [f_1^2 + f_2^2 + f_{1xx}^2 + f_{2xx}^2]\, dx\, dt \tag{53}$$

follows, where the constant M_3 is defined by functions $c_1(x,t)$, $c_2(x,t)$, and number T.

To organize the passage to the limit it is enough to have estimates (50), (52), (53) (see the proof of theorem 2); the limiting functions $u(x,t)$, $v(x,t)$ will give the solution of boundary value problem II' from desired class. □

5. Solvability of boundary value problem III'

Theorem 5.1. *Let conditions (35) and*

$$c_1(x,t) \in C^1(\overline{Q^+}),\ c_2(x,t) \in C^1(\overline{Q^+}),\ c_1(x,t) \leq 0,\ c_2(x,t) \geq 0 \tag{54}$$

hold. Then the boundary value problem III' cannot have more than one solution $\{u(x,t), v(x,t)\}$ such that $u(x,t) \in V_0^+$, $v(x,t) \in V_0^+$.

Proof. Let $f_1(x,t) \equiv 0$, $f_2(x,t) \equiv 0$, and let the $\{u(x,t), v(x,t)\}$ be a solution of boundary value problem III' with such functions $f_1(x,t)$, $f_2(x,t)$.

We multiply the equation (10) by a function $-u(x,t)$, equation (11) by $\gamma v(x,t)$, $\gamma = -\frac{\alpha}{\beta} > 0$, then we integrate the received equalities over the rectangle Q^+ and sum up the results. Integrating by parts and using the conditions (12), (15), (17), (18) and also (35), (54) we obtain the following inequality

$$\int\limits_{Q^+} [u_t^2 + v_t^2]\, dx\, dt \leq 0.$$

This inequality gives $u(x,t) \equiv 0$, $v(x,t) \equiv 0$ when $(x,t) \in Q^+$.

□

Let's investigate the solvability of boundary value problem III'. Let W_2 be linear space

$$W_2 = \{w(x,t) :\ w(x,t) \in V_0^+,\ w_{xxxxtt}(x,t) \in L_2(Q^+)\}.$$

Let these space be equipped with the norm

$$\|w\|_{W_2} = \|w\|_{V_0^+} + \|w_{xxxxtt}\|_{L_2(Q^+)}.$$

Theorem 5.2. *Let conditions (35) and*

$$c_1(x,t) \leq 0, \quad c_2(x,t) \geq 0, \quad c_{1t}(x,0) \geq 0, \quad c_{2t}(x,T) \geq 0,$$

$$c_{1xx}(x,t) \geq 0,\ c_{2xx}(x,t) \leq 0,\ T^4 \max_Q c_1^2(x,t) < 1,\ T^4 \max_Q c_2^2(x,t) < 1; \tag{55}$$

$$c_1(0,t) + c_2(0,t) = 0,\ c_{ix}(0,t) = c_{ix}(1,t) = 0,$$

$$f_i(0,t) = f_{ix}(0,t) = f_i(1,t) = 0, \quad t \in [0,T]; \tag{56}$$

$$f_i(x,t) \in L_2(Q^+), \quad f_{ixx}(x,t) \in L_2(Q^+),\ i = 1,2 \tag{57}$$

hold. Then the boundary value problem III' has a solution $(u(x,t), v(x,t))$ such that $u(x,t) \in V_0^+$, $v(x,t) \in V_0^+$.

Proof. We use the regularization method and the method of parameter continuation again.

Let's consider the boundary value problem: *find a solution* $u(x,t)$, $v(x,t)$, $(x,t) \in Q^+$ *of equations* $10''_\varepsilon$, $(11''_\varepsilon)$, *satisfying conditions (12), (15), (17), (18), (47), (48).*

We show that the boundary value problem $(10''_\varepsilon)$, $(11''_\varepsilon)$, (7), (11), (14), (15), (47), (48) is solvable in the space W_2 for any functions $f_1(x,t)$, $f_2(x,t)$ such that $f_1(x,t) \in L_2(Q^+)$, $f_2(x,t) \in L_2(Q^+)$ when ε is fixed.

Let λ be a positive number from interval $[0,1]$. We consider the family of boundary value problems: *find a solutions* $u(x,t)$, $v(x,t)$, $(x,t) \in Q^+$ *of equations* $(10''_\varepsilon)$, $(11''_\varepsilon)$ *respectively, satisfying conditions (12), (15), (17), (18), (47), (48),* (15_λ), (48_λ).

Note that, the boundary value problem $(10''_\varepsilon)$, $(11''_\varepsilon)$, (12), (15), (17), (18), (47), (48), (15_0), (48_0) has a solution in W_2, when functions $f_1(x,t)$, $f_2(x,t)$ belong to $L_2(Q^+)$ and ε is fixed. Let's show that there is a uniform, with respect to λ, estimate (49).

We multiply the equation $(10''_\varepsilon)$ by a function $-u(x,t)$, equation $(11''_\varepsilon)$ by $\gamma v(x,t)$, $\gamma = -\frac{\alpha}{\beta} > 0$. Let's integrate the received equalities over the rectangle and sum up the results. Using conditions (55), (15_λ), elementary and Young's inequalities we obtain the following inequality

$$\int\limits_{Q^+} (u_t^2 + v_t^2)\, dx\, dt \le N_1 \left\{ \varepsilon^2 \int\limits_{Q^+} [u_{xxxxt}^2 + v_{xxxxt}^2]\, dx\, dt + \int\limits_{Q^+} [f_1^2 + f_2^2]\, dx\, dt \right\}, \tag{58}$$

where the number N_1 is defined by numbers α, β, T.

Further, we multiply equation $(10''_\varepsilon)$ by a function $(-u_{xxxx}(x,t))$, equation $(11''_\varepsilon)$ by $\gamma v_{xxxx}(x,t)$, $\gamma = -\frac{\alpha}{\beta}$ and integrate the received equalities over the rectangle Q^+ and sum up the results. Integrating by parts and using the conditions ((12), (17), (18), (15_λ), (48_λ), (47) we obtain the following equality

$$\int\limits_{Q^+} [\varepsilon u_{xxxxt}^2 + \gamma\varepsilon v_{xxxxt}^2 + u_{xxt}^2 + \gamma v_{xxt}^2]\, dx\, dt = -\int\limits_{Q^+} f_1 u_{xxxx}\, dx\, dt +$$

$$+\gamma \int\limits_{Q^+} f_2 v_{xxxx}\, dx\, dt + \int\limits_{Q^+} c_1 u u_{xxxx}\, dx\, dt - \gamma \int\limits_{Q^+} c_2 v v_{xxxx}\, dx\, dt. \tag{59}$$

The terms of the right side of (59) are estimated using the Young inequality, then the received integrals — with the help of elementary integral inequalities and the inequality (58).

When (55) we obtain the following estimate

$$\varepsilon \int\limits_{Q^+} \left(u_{xxxxt}^2 + v_{xxxxt}^2\right)\, dx\, dt + \int\limits_{Q^+} \left(u_{xxt}^2 + v_{xxt}^2\right)\, dx\, dt \le N_2 \int\limits_{Q^+} \left(f_1^2 + f_2^2\right)\, dx\, dt, \tag{60}$$

where the number N_2 is defined by functions $c_1(x,t)$, $c_2(x,t)$ and numbers α, β, T, ε.

According to (58) and (60) with the help of elementary inequalities it is not difficult to deduce a desired estimate (49).

So, for a solution of boundary value problem $(10''_\varepsilon)$, $(11''_\varepsilon)$, ((12), (15), (17), (18), (47), (48), (15_λ), (48_λ) there is a uniform with respect to λ estimate (49). From this estimate and the solvability in W_2 of boundary value problem $(10''_\varepsilon)$, $(11''_\varepsilon)$, (12), (15), (17), (18), (47), (48), (15_0), (48_0) the possibility of applying the theorem about the method of parameter continuation [7] follows (see the proof of theorem 2). Therefore, the boundary value problem $(10''_\varepsilon)$, $(11''_\varepsilon)$, (12), (15), (17), (18), (47), (48) has a solution in W_2, when functions $f_1(x,t)$, $f_2(x,t)$ belong to $L_2(Q^+)$ and ε is fixed.

Let ε_0 be positive number which value will be specified below, ε is a number from the interval $(0, \varepsilon_0)$.

Let's show that for the family of solutions $\{u_\varepsilon(x,t), v_\varepsilon(x,t)\}$ of boundary value problem ($10''_\varepsilon$), ($11''_\varepsilon$), (12), (15), (17), (18), (47), (48) an a priory estimate takes place. This estimate is both uniform, with respect to parameter ε, and with its help it will be possible to organize the passage to the limit.

Note that for the family $\{u_\varepsilon(x,t), v_\varepsilon(x,t)\}$ an inequality (58) takes place.

Further, in terms of the right side of equality (59) we integrate by parts two times, and at performance (55), (56) we will receive

$$\int\limits_{Q^+} [\varepsilon u^2_{\varepsilon xxxxt} + \gamma\varepsilon v^2_{\varepsilon xxxxt} + u^2_{\varepsilon xxt} + \gamma v^2_{\varepsilon xxt} - c_1(x,t)u^2_{\varepsilon xx} + \gamma c_2(x,t)v^2_{\varepsilon xx} +$$

$$+ c_{1xx}(x,t)u^2_{\varepsilon x} - \gamma c_{2xx}(x,t)v^2_{\varepsilon x}]\,dx\,dt = -\int\limits_{Q^+} f_{1xx}u_{\varepsilon xx}\,dx\,dt + \gamma\int\limits_{Q^+} f_{2xx}v_{\varepsilon xx}\,dx\,dt +$$

$$+\int\limits_{Q^+} c_{1xx}u_\varepsilon u_{\varepsilon xx}\,dx\,dt - \gamma\int\limits_{Q^+} c_{2xx}v_\varepsilon v_{\varepsilon xx}\,dx\,dt. \qquad (61)$$

The first two terms on the right side of (61) can be estimated with the help of Young's inequality and elementary inequalities. To estimate the remaining terms we use the inequality (58).

Let the ε_0 be such that the inequality $2N_3\varepsilon < 1$ is carried out when $\varepsilon < \varepsilon_0$. The number N_3 is defined by numbers α, β, T and functions $c_1(x,t)$, $c_2(x,t)$. Then we obtain an a priory estimate for the solution of boundary value problem ($10''_\varepsilon$), ($11''_\varepsilon$), (12), (15), (17),(18), (47), (48):

$$M(\varepsilon)\int\limits_{Q^+} [u^2_{\varepsilon xxxxt} + v^2_{\varepsilon xxxxt}]\,dx\,dt + \int\limits_{Q^+} [u^2_{\varepsilon xxt} + v^2_{\varepsilon xxt}]\,dx\,dt \le M_1\int\limits_{Q^+} [f^2_{1xx} + f^2_{2xx}]\,dx\,dt, \qquad (62)$$

where the number $M(\varepsilon)$ is infinitesimal at $\varepsilon \to 0$, and the constant M_1 is defined by functions $c_1(x,t)$, $c_2(x,t)$ and number T.

To organize the passage to the limit it is enough to have estimates (58), (62) (see the proof of theorem 2), limiting functions $u(x,t)$ and $v(x,t)$ will give a solution of boundary value problem III' from a required class. □

6. Solvability of boundary value problems I–III

As mentioned above, the solvability of boundary value problems I—III is defined through the solvability of boundary value problems I'–III'.

Let's define spaces V_0^- and V_0:

$$V_0^- = \{v(x,t): \quad v(x,t) \in L_2(Q^-), \quad v_{tt}(x,t) \in L_2(Q^-),$$

$$v_{xxt}(x,t) \in L_2(Q^-)\}, \qquad V_0 = \{v(x,t): \quad v(x,t) \in V_0^+, \quad v(x,t) \in V_0^-\}.$$

Further, we denote by $c_1(x,t)$ and $f_1(x,t)$ the restriction of the function $c(x,t)$ and $f(x,t)$ to rectangle $\overline{Q^+}$, by $c_2(x,t)$ and $f_2(x,t)$ — functions $c(-x,t)$ and $f(-x,t)$ to $\overline{Q^+}$ respectively.

Statement 6.1. *Let $u(x,t)$ and $v(x,t)$ be solutions of equations (9), (10) such that $u(x,t) \in V_0^+$, $v(x,t) \in V_0^+$. Then the function $\overline{u}(x,t)$, which is defined by equality*

$$\overline{u}(x,t) = \begin{cases} u(x,t), & (x,t) \in Q^+, \\ v(-x,t), & (x,t) \in Q^- \end{cases}$$

is a solution of equation (1) *on the set* Q_1. *And it belongs to space* V_0. *Conversely, if the function* $\overline{u}(x,t)$ *is a solution of equation* (1) *on the set* Q_1 *and it is from space* V_0, *then functions* $u(x,t)$ *and* $v(x,t)$ *such that* $u(x,t) = \overline{u}(x,t)$, $v(x,t) = \overline{u}(-x,t)$ *when* $(x,t) \in Q^+$ *are solutions of equations* (9) *and* (10) *respectively.*

This statement is obvious.

The given statement and theorems 1 – 8 allow easily to receive theorems of existence and uniqueness of solutions of the boundary value problems I – III.

Theorem 6.1. *Let conditions* (19) *and* (20) *hold for functions* $c_1(x,t)$, $c_2(x,t)$ *and numbers* α, β. *Then the boundary value problem I cannot have more than one solution in space* V_0.

Theorem 6.2. *Let conditions* (19), (20) *and* (22) *hold for functions* $c_1(x,t)$, $c_2(x,t)$, $f_1(x,t)$, $f_2(x,t)$ *and numbers* α, β. *Then the boundary value problem I has a solution* $\overline{u}(x,t)$ *from in* V_0.

Theorem 6.3. *Let conditions* (32), (33) *and* (34) *hold for functions* $c_1(x,t)$, $c_2(x,t)$ *and numbers* α, β. *Then the boundary value problem II cannot have more than one solution in space* V_0.

Theorem 6.4. *Let conditions* (35) *and* (36) *hold for functions* $c_1(x,t)$, $c_2(x,t)$ *and numbers* α, β. *Then the boundary value problem II cannot have more than one solution in space* V_0.

Theorem 6.5. *Let conditions* (32), (33), (37) *and* (38) *hold for functions* $c_1(x,t)$, $c_2(x,t)$, $f_1(x,t)$, $f_2(x,t)$ *and numbers* α, β. *Then the boundary value problem II has solution* $\overline{u}(x,t)$ *in space* V_0.

Theorem 6.6. *Let conditions* (35), (44), (45) *and* (46) *hold for functions* $c_1(x,t)$, $c_2(x,t)$, $f_1(x,t)$, $f_2(x,t)$ *and numbers* α, β. *Then the boundary value problem II has solution* $\overline{u}(x,t)$ *in space* W_0. *The space* W_0 *is defined similarly to* V_0.

Theorem 6.7. *Let conditions* (35) *and* (54) *hold for functions* $c_1(x,t)$, $c_2(x,t)$ *and numbers* α, β. *Then the boundary value problem III cannot have more than one solution in space* V_0.

Theorem 6.8. *Let conditions* (35), (55), (56) *and* (57) *hold for functions* $c_1(x,t)$, $c_2(x,t)$, $f_1(x,t)$, $f_2(x,t)$ *and numbers* α, β. *Then the boundary value problem III has solution* $\overline{u}(x,t)$ *in space* V_0.

7. Comments and additions

1. For problems I and II conditions (19) and (32) of theorems 6 — 8, 10 hold, for example, when natural sewing conditions

$$u(-0,t) = u(+0,t), \quad u_x(-0,t) = u_x(+0,t)$$

are given. But the condition (35) of theorem 9 holds when discontinuous sewing conditions

$$u(-0,t) = u(+0,t), \quad u_x(-0,t) = -u_x(-0,t)$$

or

$$u(-0,t) = -u(+0,t), \quad u_x(-0,t) = u_x(+0,t)$$

are given.

2. The equation (1), system of equations (10) and (11) can include the lowest terms, moreover the system of equations (10) and (11) can be bound, for example

$$u_{tt} - u_{xxt} + c_1(x,t)u + b_1(x,t)v = f_1(x,t),$$

$$-v_{tt} - v_{xxt} + c_2(x,t)v + b_2(x,t)u = f_2(x,t).$$

The conditions on the lowest terms can be written out and the received results for related system will also have independent significance.

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