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CONTROL OF MOVABLE SOURCES IN DISTRIBUTED SYSTEMS ON THE CLASSES OF IMPULSIVE, PIECEWISE CONSTANT, AND HEAVISIDE FUNCTIONS

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Abstract. Problems of optimal control of movable lumped sources in distributed systems when controls belong to the classes of impulsive, piecewise constant, and Heaviside functions are considered in the paper. The optimal control problems are investigated for various cases, according to the position of the sources. Necessary optimality conditions are obtained for optimal control problems considered on all these classes. Constructive analytical formulas for the gradient of a functional in the space of optimized parameters are derived. The results of some numerical experiments are given.

Keywords: impulsive control, piecewise constant control, Heaviside function, gradient of functional, movable lumped sources.

AMS Subject Classification: 49K, 49M, 65K, 65N.

1. Introduction

The interest to problems of motions of the sources in one or another meaning, with various physical natures has increased lately. For example, they arise when investigating movable sources of a chemical agent's concentration, underground waters' mass (pressure), oil (generally, of substance), impulse, tension, thermo tension, heat, voice, radiation, electromagnetic waves' emission (generally, energy), information, etc. Similar problems also arise when solving some inverse problems of mathematical physics.

Along with continuous displacements of a source, there may be instances when the source can move from one position to another only unevenly, and an optimal movable control is only to be found on the class of such step-wise displacements.

When controlling real-life objects, the optimization of control actions on the classes of continuous and piecewise continuous functions causes some technical difficulties. The solution to optimal control problems on the classes of functions technically easily implemented is of important value [1]-[8],[11]-[13]. Systems with control actions from the classes of impulsive, piecewise constant, and, particularly, Heaviside functions can be related to such classes. In contrast to a lot of other works [1], [3]-[5], [11], [13] here the control actions (factors) are not only the sources' power, but also the positions and moments of their application.

Similar optimal control problems on the class of impulsive functions are considered in [1], [5], [13] for ordinary differential equations. In [2], the problem of oil wells' placement and of control of oil output is considered.

In the mathematical models of a lot of controlled processes, Heaviside step functions are used as control actions. It is clear that this is a particular case of a piecewise constant function, but the control in the form of Heaviside functions is of interest from the practical point of view, since in practice, a lot of controlled processes are such that every action takes on a value that is constant in time and is switched on only once.

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The problems of optimal control of the sources' motion and of their intensity (power) are being investigated by a lot of authors on the class of piecewise continuous functions [3], [7]. The statement of the problem of control of lumped sources for two dimensional cases is investigated in the work [3]; in this case, the optimization consists in determining an optimal law of the sources' motion (trajectories), as well as their power.

The problems of optimal control of lumped sources in distributed systems when the controls belong to the classes of impulsive, piecewise constant, and Heaviside functions are considered in the paper. The optimal control problems are investigated for various cases, according to the control of the sources' position.

Analytical formulas for the gradient of a functional in the space of optimized parameters for the optimal control problems on these classes are derived. These formulas allow one to use the first order optimization methods [15] to find the numerical solution to the optimal control problem. The results of numerical experiments are given.

2. Problem statement

Consider the following optimal control problem for systems with distributed parameters, which consists in minimizing the functional

$$J(w)=\alpha_1\int\limits_{\Omega}[u(x,T;w)-U(x)]^2dx+\alpha_2\left\|v(t)-v^0(t)\right\|^2_{L_2}+\alpha_3\left\|s(t)-s^0(t)\right\|^2_{L_2}. \quad (1)$$

The state of the controlled object is described by the following n - dimensional boundary problem of parabolic type:

$$u_t(x,t)=\ div\,(\sigma(x)grad\,u(x,t))+\sum_{i=1}^{L}v_i(t)b_i(x,t)\delta(x-\xi^i(t)),x\in\Omega\subset R^n,0<t\le T, \quad (2)$$

$$u(x,0)=\varphi(x),x\in\Omega, \quad (3)$$

$$u(x,t)|_{x\in\Gamma^1}=\mu_1(x,t),\sigma(x)\frac{\partial u(x,t)}{\partial n}|_{x\in\Gamma^2}=\mu_2(x,t),0<t\le T, \quad (4)$$

$$\Gamma=\Gamma^1\bigcup\Gamma^2=\partial\Omega,\Gamma^1\bigcap\Gamma^2=\emptyset.$$

Here $\frac{\partial u(x,t)}{\partial n}=\sum\limits_{i=1}^{n}u_{x_i}\cos(n\wedge e_i)$; n is the unique internal normal to the part Γ^2 of the domain boundary; $u=u(x,t)=u(x,t;w)$ the phase state of the object determined from the solution to boundary problem (2)–(4) on the corresponding admissible value of the optimized control vector $w=(v(t),s(t))$; R^n -n-dimensional Euclidian space; L-given number of control actions (sources); $b_1(x,t),...,b_L(x,t)$, $\varphi(x)$, $\mu_1(x,t),\mu_2(x,t)$, $\sigma(x)$, $U(x)$, $\alpha_i>0$, $i=1,2$, $l>0$, $T>0$, $v^0(t)=(v^0_1(t),....,v^0_L(t))$, $s^0(t)=(s^0_1(t),....,s^0_L(t))$ are given continuous functions and values determining the investigated process and the criterions of control on it; $\delta(x)=\prod_{i=1}^{n}\delta(x_i)$, $\delta(x_i)$ generalized Dirac function; $v(t)=(v_1(t),....,v_L(t))$ the controls determining the sources' power; $\xi^i(t)\in R^n$ the coordinates of i-th source's placement at the point of time t, $\xi(t)=(\xi^1(t),....,\xi^L(t))$; e_i-i-th ort coordinate.

With respect to the sources' placement $\xi^i(t)=(\xi^i_1(t),...,\xi^i_n(t))$, $i=1,...,L$, we can consider the following variants.

The sources are motionless:

$$\xi^i(t)=\xi^i=const,\ t\in[0,T],\xi^i\in R^n,i=1,...,L,$$

or the sources are movable and their motion law is determined by a Cauchy problem with respect to the following systems of differential equations

$$\dot{\xi}^i(t) = f^i(\xi^i, s_i(t), t), \xi^i(0) = \xi_0^i, t \in [0, T], i = 1, ..., L, \tag{5}$$

where $\xi_0^i \in R^n$, $i = 1, ..., L$ are given initial values of the sources' placement; $f^i = f^i(\cdot, \cdot; \cdot)$, $i = 1, ..., L$ given n dimensional vector-functions; $s_i(t)$, $i = 1, ..., L$ m_i-dimensional control actions on the source's motion. Thus the sources' mechanical motions can be controlled processes as well.

The problems of optimal control of the sources where the object (process) is controlled by the sources' intensity (power) and motion are investigated in the work. We also consider the problems when the moments of the sources' actions, as well as the coordinates ξ^i, $i = 1, ..., L$ of the motionless sources' placement are optimized.

Optimal control problems (1)-(5) are considered for the following classes of control actions:

a) The sources' controlled powers are from the class of impulsive functions

$$v_i(t) = \sum_{j=1}^{m_i} q_{ij}\delta(t - \theta_{ij}), M = \sum_{i=1}^{L} m_i, i = 1, .., L, \tag{6}$$

and determined by finite-dimensional vector w $= (q, \theta) \in R^{2LM}$, where q_{ij} is the value of the impulsive power of the $i-$th source at the points of time θ_{ij}, $j = 1, .., m_i$, $i = 1, .., L$; m_i given number of impulsive actions of the $i-$th source, i.e. functional (1) is determined by the finite-dimensional vector:

$$w = (q_{11}, q_{12}, ..., q_{1m_1}, ..., q_{Lm_L}, \theta_{11}, \theta_{12}, ..., \theta_{1m_1}, ..., \theta_{Lm_L}). \tag{7}$$

Consider the following constraints on the control parameters:

$$\underline{q_i} \le q_{ij} \le \overline{q_i},\ 0 \le \zeta < \theta_{ij} - \theta_{i-1j} \le \eta,\ \theta_{ij} \in [0, T],\ j = 1, ..., m_i,\ i = 1, ..., L, \tag{8}$$

where $\underline{q_j}, \overline{q_j}, \zeta, \eta$ are given.

b) The sources' controlled powers are from the class of piecewise constant functions

$$v_i(t) = q_{ij} = const, \quad t \in [\theta_{ij-1}, \theta_{ij}), \theta_{ij-1} < \theta_{ij},\ j = 1, ...m_i, \tag{9}$$

$$\theta_{i0} = 0, \quad \theta_{im} = T, \quad M = \sum_{i=1}^{L} m_i, i = 1, ..., L,$$

and determined by finite-dimensional vector $w = (q, \theta) \in R^{2LM}$, i.e. the values of controls $v_i(t)$ are constant on semi-intervals $[\theta_{ij-1}, \theta_{ij}) \subset [0, T]$ and belong to some admissible set V, particularly, to parallelepiped (8), and θ_{ij}, $j = 1, ..., m_i - 1$, $i = 1, ..., L$ are determined on the intervals of constancy $[\theta_{ij-1}, \theta_{ij})$ of the value of the i-th source's power, m_i given number of constancy intervals for the $i-$th source.

c) The sources' controlled powers are from the class of Heaviside functions

$$v_i(t) = v_i(t; q_i, \theta_i) = q_i\chi(t - \theta_i),\ i = 1, ..., L, \tag{10}$$

and determined by finite-dimensional vector

$$w = (q, \theta) = (q_1, ..., q_L, \theta_1, ..., \theta_L) \in R^{2L}, \tag{11}$$

where the $i-$th component is the power of the $i-$th source beginning to influence at the point of time w$_{L+i} = \theta_i$, $i = 1, .., L$; $\chi(t - \theta_i)$ is Heaviside function.

Consider the following constraints on the control parameters:

$$\underline{q_i} \le q_i \le \overline{q_i},\ 0 \le \theta_i \le T,\ i = 1, ..., L. \tag{12}$$

Here $\underline{q_i}, \overline{q_i}$, L are given. Each component of the control vector-function (control) $v(t)$ is piecewise constant function with only one change of value and is determined by the values θ_i and q_i, i.e. by the control's action time and value.

d) The source's motion, which is described by system of differential equations (5), is implemented by a control from the class of impulsive functions

$$s_i(t) = \sum_{j=1}^{m_i} s_{ij}\delta(t - \tau_{ij}), M = \sum_{i=1}^{L} m_i, i = 1, .., L, \tag{13}$$

and determined by finite-dimensional vector $w = (s, \tau) \in R^{2LM}$, where s_{ij} is the value of the impulsive power's influence on the motion of the $i-$th source at the point τ_{ij}, $j = 1, .., m_i$, $i = 1, .., L$; m_i given number of impulsive influences, i.e.

$$w = (s_{11}, s_{12}, ..., s_{1m_1}, ..., s_{Lm_L}, \tau_{11}, \tau_{12}, ..., \tau_{1m_1}, ..., \tau_{Lm_L}). \tag{14}$$

The constraints in form (8) can be imposed on the control parameters in the control problem.

e) The control actions $s(t)$ on the sources' motion, which is described by system of differential equations (5), are piecewise constant functions, i.e.

$$s_i(t) = s_{ij} = const,\ t \in [\tau_{ij-1}, \tau_{ij}), \tau_{ij-1} < \tau_{ij},\ j = 1, ...m_i\ , \tag{15}$$

$$\tau_{i0} = 0\ ,\ \tau_{im_L} = T,\ M = \sum_{i=1}^{L} m_i, i = 1, ..., L.$$

Thus the control is determined by the finite-dimensional vector $w = (s, \tau) \in R^{2LM}$, i.e. the values of the control actions on the trajectory are constant on semi-intervals $[\tau_{ij-1}, \tau_{ij}) \subset [0, T]$ and belong to some admissible set, for example, to (8).

f) The control actions on the sources' motion are from the class of Heaviside functions

$$s_i(t) = s_i\chi(t - \tau_i),\ \ i = 1, ..., L,$$

i.e. determined by the finite-dimensional vector

$$w = (s,\ \tau) = (s_1, ..., s_L, \tau_1, ..., \tau_L) \in R^{2L}. \tag{16}$$

Here $i-$th component of vector (16) is the power of the $i-$th control action on the source's motion. We have the constraints on the control parameters in form (13).

Assume that the functions and parameters satisfy all the existence and uniqueness conditions imposed on the solution to the boundary problem in all the optimal control problems mentioned above.

The optimal control problems considered are equivalent to the problems of the optimization of the functional $J(w)$ in an admissible closed domain; thus the set of optimal solutions is non-empty [11].

The following theorem holds true.

Theorem 2.1. *If the functional $J(w)$ is convex on the class of piecewise continuous control functions, then it is also convex on the classes of impulsive, piecewise constant, and Heaviside functions.*

The controls may be discontinuous in the considered problems, so there isn't any classical solution to these problems.

We take the function $u(x, t) = u(x, t; w)$ from the space $L_2(\Omega \times [0, T])$ under the generalized solution to the boundary problem (2)-(4) with respect to the control $w = (v(t), s(t))$ from the Hilbert space $H = L_2([0, T])$. This function satisfies the following equality

$$\int_\Omega u(x,T)\psi(x,T)dx - \int_\Omega \varphi(x)\psi(x,0)dx - \int_0^T\int_\Omega u(x,t)(\psi_t(x,t) + div(\sigma(x)grad\psi(x,t)))dxdt-$$
$$-\sum_{i=1}^L \int_0^T\int_\Omega \psi(x,t)v_i(t)b_i(x,t)\delta(x-\xi^i(t))dxdt = 0,$$

for every $\psi = \psi(x,t) \in H^{2,1}(\Omega \times [0,T])$ such that $\frac{\partial\psi(x,t)}{\partial n}|_{x\in\Gamma} = 0$ [11].

3. Numerical approach to the solution to the considered problems

Using the first order iteration optimization methods based on the application of analytical formulas for the gradient of the target functional for numerical solution to the problem of optimal control of the lumped sources' power, placement, and motion in distributed systems. For example, we suggest the use of the gradient projection methods

$$w^{k+1} = \Pr_V(w^k - \alpha_k gradJ(w^k)),\ k = 0,1,...$$

or interfaced gradient projection methods [14], [15]. Here w^0 is some given initial value of the control; $gradJ(w)$ the gradient vector of the target functional on the optimized parameters; α_k the value of one-dimensional step along the direction of the anti-gradient of the functional; $\Pr_V(\cdot)$ the operator of projection on admissible set of controls V (this operator has a simple form for positional constraints in form (15) [14]).

The formulas for the gradient of the functional obtained below can also be used to formulate necessary optimality conditions (in the form of maximum principle in the variation form) for the problems considered above. It is evident that the classes of control functions investigated in this work are the special cases of general classes considered in other works [9]-[11], [14], [15], where the formulas for the gradient of the functional are obtained in more general form. The formulas for the gradient of the functional corresponding to the specified classes of functions are obtained in this work by using the general approach for these classes of functions.

Taking into account that the control actions on the sources' power and motion are mutually independent in the problems considered, we can obtain formulas for the corresponding components of the gradient independently (what is done below).

3.1. **Formulas for the components of the gradient of the functional on the sources' power and on points of time of their actions.** Let us show that functional (1) in optimal control problem (1)-(4) is differentiable in H. For this purpose, we take two arbitrary admissible controls $w = (v(t), s(t))$ and $w + \Delta w = (v(t) + \Delta v(t), s(t))$. Let $u(x,t;w)$, $u(x,t;w+\Delta w)$ be the solutions to the boundary problem (2)-(4) corresponding to these controls. Introduce the notation

$$\Delta u(x,t) = u(x,t;w+\Delta w) - u(x,t;w).$$

From (2)-(4), it follows that $\Delta u(x,t)$ is a generalized solution to the following boundary problem:

$$\Delta u_t(x,t) = div\,(\sigma(x)grad\,\Delta u(x,t)) + \sum_{i=1}^L \Delta v_i(t)b_i(x,t)\delta(x-\xi^i(t)), x\in\Omega\subset R^n, 0<t\le T, \quad (17)$$

$$\Delta u(x,0) = 0, x\in\Omega, \quad (18)$$

$$\Delta u(x,t)|_{x\in\Gamma^1} = 0, \sigma(x)\frac{\partial \Delta u(x,t)}{\partial n}|_{x\in\Gamma^2} = 0, 0 < t \le T. \tag{19}$$

Then the increment of functional (1) can be written as follows

$$\Delta J(w) = J(w+\Delta w) - J(w) = 2\alpha_1 \int_\Omega [u(x,T;w) - U(x)]\Delta u(x,T)dx +$$
$$+\alpha_1 \int_\Omega |\Delta u(x,T)|^2 dx + \alpha_2(\left\|v(t)+\Delta v(t) - v^0(t)\right\|^2 - \left\|v(t) - v^0(t)\right\|^2).$$

Let $\psi(x,t)$ be the solution to the following conjugate problem ([11], [14]):

$$\psi_t(x,t) + div(\sigma(x) grad\psi(x,t)) = 0, x\in\Omega, 0<t\le T, \tag{20}$$

$$\psi(x,T) = 2\alpha_1(u(x,T) - U(x)),\ x\in\Omega, \tag{21}$$

$$\psi(x,t)\,|_{x\in\Gamma^1} = 0, \sigma(x)\frac{\partial\psi(x,t)}{\partial n}|_{x\in\Gamma^2} = 0, 0<t\le T. \tag{22}$$

From (17)-(22), it follows that

$$2\alpha_1 \int_\Omega (u(x,T) - U(x))\Delta u(x,T)dx = \int_\Omega \psi(x,T)\Delta u(x,T)dx = \int_\Omega \int_0^T \frac{\partial}{\partial t}(\psi(x,t)\Delta u(x,t))dt)dx =$$
$$= \int_0^T \int_\Omega (\psi_t(x,t)\Delta u(x,t) + \psi(x,t)\Delta u_t(x,t))dxdt =$$
$$= \int_0^T \int_\Omega (-\Delta u(x,t) div(\sigma(x) grad\psi(x,t)) + \psi(x,t) div(\sigma(x) grad\Delta u(x,t)))dxdt +$$
$$+ \sum_{i=1}^L \int_0^T \int_\Omega \psi(x,t) b_i(x,t)\Delta v_i(t)\delta(x-\xi^i(t))dxdt =$$
$$= \int_0^T \int_\Omega \sigma(x)(\psi(x,t) div(\Delta u(x,t)) - \Delta u(x,t) div(\psi(x,t)))dxdt +$$
$$+ \sum_{i=1}^L \int_0^T \int_\Omega \psi(x,t) b_i(x,t)\Delta v_i(t)\delta(x-\xi^i(t))dxdt =$$
$$= \int_0^T \int_\Gamma \sigma(x)(\psi(x,t)\frac{\partial\Delta u(x,t)}{\partial n} - \Delta u(x,t)\frac{\partial\psi(x,t)}{\partial n})dsdt +$$
$$+ \sum_{i=1}^L \int_0^T \int_\Omega \psi(x,t) b_i(x,t)\Delta v_i(t)\delta(x-\xi^i(t))dxdt.$$

By using the estimation obtained in [10], [14] for the more general case, and taking controls from the class of measurable functions

$$\int_{\Omega} |\Delta u(x,T)|^2 dx \le C \int_0^T |\Delta v(t)|^2 dt, \tag{23}$$

where $C > 0$ is constant independent on the choice of Δv, we have the following formula for the increment of the functional (1)

$$\begin{aligned}\Delta J(w) = \sum_{i=1}^{L} \int_0^T \int_{\Omega} \psi(x,t) b_i(x,t) \Delta v_i(t) \delta(x - \xi^i(t)) dx dt + \\ +2\alpha_2 \int_0^T (v_i(t) - v_i^0(t)) \Delta v(t) dt + o\left(\|\Delta v(t)\|\right) .\end{aligned} \tag{24}$$

Next, we consider particular cases of (24) when obtaining formulas for the gradient of the functional in the space of the optimized parameters for all the classes of functions mentioned above.

3.1.1. *Sources' controlled powers are from the class of impulsive functions.* Obtain formulas for the gradient $\frac{dJ(w)}{dq_{ij}}$, $j = 1, ..., m_i$, $i = 1, ..., L$ of the functional using (24). The increment of the functional for the increment of the argument $q_{ij}$of the vector w from (6) by Δq_{ij}, i.e. $\Delta w = (\Delta_{ij} q, 0) \in R^{2LM}$, $\Delta_{ij} q = (0, ..., \Delta q_{ij}, ..., 0) \in R^{ML}$, can be written as follows:

$$\begin{aligned}\Delta_{q_{ij}} J(w) = \int_0^T \int_{\Omega} \psi(x,t) b_i(x,t) \Delta q_{ij} \delta(t - \theta_{ij}) \delta(x - \xi^i(t)) dx dt + \\ +2\alpha_2 \int_0^T (v_i(t) - v_i^0(t)) \Delta q_{ij} \delta(t - \theta_{ij}) dt + o\left(\|\Delta v(t)\|\right) = \\ = \psi(\xi^i(\theta_{ij}), \theta_{ij}) b_i(\xi^i(\theta_{ij}), \theta_{ij}) \Delta q_{ij} + 2\alpha_2 (v_i(\theta_{ij}) - v_i^0(\theta_{ij})) \Delta q_{ij} + o(\|\Delta_{ij} q\|).\end{aligned}$$

Dividing both parts into Δq_{ij} and proceeding to the limit as $\Delta q_{ij} \to 0$, we have:

$$\frac{dJ(w)}{dq_{ij}} = \psi(\xi^i(\theta_{ij}), \theta_{ij}) b_i(\xi^i(\theta_{ij}), \theta_{ij}) + 2\alpha_2 (v_i(\theta_{ij}) - v_i^0(\theta_{ij})), i = 1, ..., L. \tag{25}$$

(25) is the formula for the components of the gradient of the functional on the impulsive power in problem (1)–(8).

Now obtain the formulas for derivatives $\frac{dJ(w)}{d\theta_{ij}}$, $j = 1, ..., m_i$, $i = 1, ..., L$. For this purpose, introduce the following function [8]

$$\delta_\varepsilon(t - \tau) = \begin{cases} \frac{1}{\varepsilon}, & t \in [\tau - \varepsilon, \tau], \\ 0, & t \notin [\tau - \varepsilon, \tau], \end{cases} \tag{26}$$

where $\varepsilon > 0$, τ are parameters; t the function's argument. It is obvious that when ε tends to zero the function $\delta_\varepsilon(\cdot)$ approaches the generalized Dirac function $\delta(\cdot)$. Increment τ by $\Delta\tau = \varepsilon$. Then δ_ε obtains the following increment

$$\Delta_\tau \delta_\varepsilon(t - \tau) = \begin{cases} 0, & t \notin [\tau - \varepsilon, \tau + \Delta\tau], \\ -1/\varepsilon, & t \in [\tau - \varepsilon, \tau + \Delta\tau - \varepsilon], \\ 1/\varepsilon, & t \in [\tau + \Delta\tau - \varepsilon, \tau + \Delta\tau]. \end{cases} \tag{27}$$

The increment of δ_ε will have the following form when $\Delta\tau < \varepsilon$:

$$\Delta_\tau \delta_\varepsilon(t-\tau) = \begin{cases} 0, & t \notin [\tau-\varepsilon, \tau+\Delta\tau], t \in [\tau+\Delta\tau-\varepsilon, \tau], \\ -1/\varepsilon, & t \in [\tau-\varepsilon, \tau+\Delta\tau-\varepsilon], \\ 1/\varepsilon, & t \in [\tau, \tau+\Delta\tau]. \end{cases} \tag{28}$$

We have then for $\Delta\tau > \varepsilon$

$$\Delta_\tau \delta_\varepsilon(t-\tau) = \begin{cases} 0, & t \notin [\tau-\varepsilon, \tau+\Delta\tau], t \in [\tau, \tau+\Delta\tau-\varepsilon], \\ -1/\varepsilon, & t \in [\tau-\varepsilon, \tau], \\ 1/\varepsilon, & t \in [\tau+\Delta\tau-\varepsilon, \tau+\Delta\tau]. \end{cases} \tag{29}$$

If we increment the argument θ_{ij} of the vector w from (7) by $\Delta\theta_{ij}$, i.e. $\Delta w = (0, \Delta_{ij}\theta) \in R^{2LM}$, $\Delta_{ij}\theta = (0, ..., \Delta\theta_{ij}, ..., 0) \in R^{LM}$, then we have the increment of the functional in the following form by using (24)

$$\begin{gathered} \Delta_{\theta_{ij}} J(w) = \int_0^T \int_\Omega \psi(x,t) b_i(x,t) q_{ij} \Delta_{\theta_{ij}} \delta_\varepsilon(t-\theta_{ij}) \delta(x-\xi^i(t)) dx dt + \\ +2\alpha_2 \int_0^T (q_{ij}\delta(t-\theta_{ij}) - q_i^0 \delta_\varepsilon(t-\theta_{ij}^0)) q_{ij} \Delta_{\theta_{ij}} \delta_\varepsilon(t-\theta_{ij}) dt + o(\|\Delta v(t)\|). \end{gathered} \tag{30}$$

Taking (27) into account, from (30), we obtain the following formula for $\Delta\theta_{ij} = \varepsilon$

$$\begin{gathered} \Delta_{\theta_{ij}} J(w) = \frac{q_{ij}}{\varepsilon} \int_\Omega \left(\int_{\theta_{ij}+\Delta\theta_{ij}-\varepsilon}^{\theta_{ij}+\Delta\theta_{ij}} \psi(x,t) b_i(x,t) dt - \int_{\theta_{ij}-\varepsilon}^{\theta_{ij}+\Delta\theta_{ij}-\varepsilon} \psi(x,t) b_i(x,t) dt \right) \times ly \\ \times \delta(x-\xi^i(t)) dx + +2\alpha_2 q_{ij} \int_{\theta_{ij}+\Delta\theta_{ij}-\varepsilon}^{\theta_{ij}+\Delta\theta_{ij}} (q_{ij}\delta(t-\theta_{ij}) - q_{ij}^0 \delta_\varepsilon(t-\theta_{ij}^0)) dt - \\ - \int_{\theta_{ij}-\varepsilon}^{\theta_{ij}+\Delta\theta_{ij}-\varepsilon} (q_{ij}\delta(t-\theta_{ij}) - q_{ij}^0 \delta_\varepsilon(t-\theta_{ij}^0)) dt + o(\|\Delta v(t)\|) = \\ = \frac{q_{ij}}{\varepsilon} \int_\Omega \left(\int_{\theta_{ij}-\varepsilon}^{\theta_{ij}} \psi(x, t+\Delta\theta_{ij}) b_i(x, t+\Delta\theta_{ij}) - \psi(x,t) b_i(x,t) dt \right) \delta(x-\xi^i(t)) dx + \\ +o(\|\Delta v(t)\|). \end{gathered} \tag{31}$$

Taking into account (29), for the case $\Delta\theta_{ij} > \varepsilon$, we have

$$\begin{aligned}\Delta_{\theta_{ij}} J(w) &= \frac{q_{ij}}{\varepsilon}\int\limits_{\Omega}\left(\int\limits_{\theta_{ij}+\Delta\theta_{ij}-\varepsilon}^{\theta_{ij}+\Delta\theta_{ij}} \psi(x,t)b_i(x,t)dt - \int\limits_{\theta_{ij}-\varepsilon}^{\theta_{ij}} \psi(x,t)b_i(x,t)dt\right)\times \\ &\times\delta(x-\xi^i(t))dx + 2\alpha_2 q_i \int\limits_{\theta_{ij}+\Delta\theta_{ij}-\varepsilon}^{\theta_{ij}+\Delta\theta_{ij}} (q_{ij}\delta(t-\theta_{ij}) - q_{ij}^0\delta_\varepsilon(t-\theta_{ij}^0))dt- \\ &- \int\limits_{\theta_{ij}-\varepsilon}^{\theta_{ij}} (q_{ij}\delta(t-\theta_{ij}) - q_{ij}^0\delta_\varepsilon(t-\theta_{ij}^0))dt + o(\|\Delta v\|) = \\ &= \frac{q_{ij}}{\varepsilon}\int\limits_{\Omega}\left(\int\limits_{\theta_{ij}-\varepsilon}^{\theta_{ij}} \psi(x,t+\Delta\theta_{ij})b_i(x,t+\Delta\theta_{ij}) - \psi(x,t)b_i(x,t)dt\right)\times \\ &\times\delta(x-\xi^i(t))dx + o(\|\Delta v\|).\end{aligned} \tag{32}$$

Expanding the function $\psi(x,t)b_i(x,t)$ into Taylor series within the neighborhood of t, we have:

$$\begin{aligned}&\int\limits_{\theta_{ij}-\varepsilon}^{\theta_{ij}} [\psi(x,t+\Delta\theta_{ij})b_i(x,t+\Delta\theta_{ij}) - \psi(x,t)b_i(x,t)]dt = \\ &= \int\limits_{\theta_{ij}-\varepsilon}^{\theta_{ij}} (\psi(x,t)b_i(x,t))'_t\Delta\theta_{ij}dt + o(\Delta\theta_{ij}) = \Delta\theta_{ij}(\psi(x,t)b_i(x,t))\big|_{\theta_{ij}-\varepsilon}^{\theta_{ij}} + o(\Delta\theta_{ij}).\end{aligned} \tag{33}$$

Taking (33) into account in (31), (32), we get

$$\Delta_{\theta_{ij}} J(w) = \frac{q_{ij}}{\varepsilon}\int\limits_{\Omega} (\Delta\theta_{ij}(\psi(x,t)b_i(x,t) - \psi(x,t-\varepsilon)b_i(x,t-\varepsilon)))\,|_{t=\theta_{ij}}\delta(x-\xi^i(t))dx + o(\Delta\theta_{ij}),$$

Dividing both parts of this equality into $\Delta\theta_{ij}$ and proceeding to the limit when $\Delta\theta_{ij} \to 0$, $\varepsilon \to 0$, we obtain

$$\frac{dJ(w)}{d\theta_{ij}} = q_{ij}(\psi(\xi^i(t),t)b_i(\xi^i(t),t))'_t|_{t=\theta_{ij}}, j = 1,...,m_i, i = 1,...,L. \tag{34}$$

Taking into account (30), from (28), we have the following formula for $\Delta\theta_{ij} < \varepsilon$

$$\begin{aligned}\Delta_{\theta_{ij}} J(w) &= \frac{q_{ij}}{\varepsilon}\int\limits_{\Omega}\left(\int\limits_{\theta_{ij}}^{\theta_{ij}+\Delta\theta_{ij}} \psi(x,t)b_i(x,t)dt - \int\limits_{\theta_{ij}-\varepsilon}^{\theta_{ij}+\Delta\theta_{ij}-\varepsilon} \psi(x,t)b_i(x,t)dt\right)\times \\ &\times\delta(x-\xi^i(t))dx + 2\alpha_2 q_{ij}\left(\int\limits_{\theta_{ij}}^{\theta_{ij}+\Delta\theta_{ij}} (q_{ij}\delta(t-\theta_{ij}) - q_i^0\delta_\varepsilon(t-\theta_{ij}^0))dt-\right. \\ &- \int\limits_{\theta_{ij}-\varepsilon}^{\theta_{ij}+\Delta\theta_{ij}-\varepsilon} (q_{ij}\delta(t-\theta_{ij}) - q_i^0\delta_\varepsilon(t-\theta_{ij}^0))dt + o(\|\Delta v(t)\|) = \\ &= \frac{q_{ij}}{\varepsilon}\int\limits_{\Omega}\left(\int\limits_{\theta_{ij}}^{\theta_{ij}+\Delta\theta_{ij}} (\psi(x,t)b_i(x,t) - \psi(x,t-\varepsilon)b_i(x,t-\varepsilon))dt\right)\times \\ &\times\delta(x-\xi^i(t))dx + o(\|\Delta v(t)\|).\end{aligned} \tag{35}$$

Expanding the function $\psi(x,t)b_i(x,t)$ into Taylor series within the neighborhood of t, and dividing both sides of (35) into $\Delta\theta_{ij}$, then proceeding to the limit when $\Delta\theta_{ij} \to 0$, $\varepsilon \to 0$, we have

$$\frac{dJ(w)}{d\theta_{ij}} = q_{ij} \int\limits_{\Omega} \lim_{\varepsilon\to 0} \lim_{\Delta\theta_{ij}\to 0} \frac{\psi(x,t+\Delta\theta_{ij})b_i(x,t+\Delta\theta_{ij}) - \psi(x,t)b_i(x,t)}{\Delta\theta_{ij}}|_{t=\theta_{ij}} \delta(x-\xi^i(t))dx$$

Here we have a coincidence with formula (34), which determines the components of the gradient of the functional with respect to the moments of impulsive influences in the problem (1)–(8).

3.1.2. *Sources' controlled powers are from the class of piecewise constant functions.* Use formula (24) for the functional's increment, and choose the increment $\Delta w = \Delta v(t)$ in the following form to obtain the formulas for the components of the gradient of the functional: $\frac{dJ(w)}{dq_{ij}}$, $j = 1, ..., m_i$, $i = 1, ..., L$ in problem (1)-(4), (9),(10), when controlling piecewise constant functions

$$\Delta v(t) = \begin{cases} 0, & 0 < t < \theta_{ij-1},\ \theta_{ij} < t < T, \\ \Delta q_{ij} = const, & \theta_{ij-1} \le t < \theta_{ij},\ j = 1, ..., m_i,\ i = 1, ..., L. \end{cases} \quad (36)$$

If to take into account (36) in (24), then the formula for the functional's increment will be as follows

$$\Delta_{q_{ij}} J(w) = \int\limits_0^T \int\limits_{\Omega} \psi(x,t)b_i(x,t)\Delta q_{ij}\delta(x-\xi^i(t))dxdt +$$
$$+2\alpha_2 \int\limits_{\theta_{ij-1}}^{\theta_{ij}} (v_i(t) - v_i^0(t))\Delta q_{ij}dt + o(\|\Delta v(t)\|) = \int\limits_{\theta_{ij-1}}^{\theta_{ij}} \psi(\xi^i(t),t)b_i(\xi^i(t),t)\Delta q_{ij}dt +$$
$$+2\alpha_2 \int\limits_{\theta_{ij-1}}^{\theta_{ij}} (v_i(t) - v_i^0(t))\Delta q_{ij}dt + o(\|\Delta v(t)\|).$$

Dividing both sides of this expression into Δq_{ij}, proceeding to the limit when $\Delta q_{ij} \to 0$, and taking into account that $o(||\Delta v(t)||)/\Delta q_{ij} \to 0$, $\Delta q_{ij} \to 0$, we have

$$\frac{dJ(w)}{dq_{ij}} = \int\limits_{\theta_{ij-1}}^{\theta_{ij}} \psi(\xi^i(t),t)b_i(\xi^i(t),t)dt + 2\alpha_2 \int\limits_{\theta_{ij-1}}^{\theta_{ij}} (v(t) - v^0(t))dt,$$

$$j = 1, ..., m_i, i = 1, ..., L. \quad (37)$$

Increment the argument θ_{ij} of the vector $w = (q,\theta)$ by $\Delta\theta_{ij}$ to obtain the formulas for the derivatives $\frac{dJ(w)}{d\theta_{ij}}$, $j = 1, ..., m_i - 1$, $i = 1, ..., L$, i.e.

$$\Delta w = (0, \Delta\theta) \in R^{2LM}, \Delta_{ij}\theta = (0, ..., \Delta\theta_{ij}, ..., 0) \in R^{L(M-L)}.$$

Assume that $\Delta\theta_{ij} > 0$. Then it is evident that the control $v(t)$ gets an increment $\Delta v(t)$ in the following form

$$\Delta v(t) = \begin{cases} 0, & 0 < t < \theta_{ij},\ \theta_{ij} + \Delta\theta_{ij} < t < T, \\ q_{ij} - q_{ij+1}, & \theta_{ij} \le t < \theta_{ij} + \Delta\theta_{ij}. \end{cases}$$

Then the functional gets an increment as

$$\Delta_{\theta_i} J(w) = \int\limits_{\theta_{ij}}^{\theta_{ij}+\Delta\theta_{ij}} (q_{ij} - q_{ij+1})\psi(\xi^i(t), t) b_i(\xi^i(t), t) dt +$$
$$+2\alpha_2 \int\limits_{\theta_{ij}}^{\theta_{ij}+\Delta\theta_{ij}} (v(t) - v^0(t))(q_{ij} - q_{ij+1}) + o(\|\Delta v(t)\|).$$

Using the average theorem, we have the formula

$$\Delta_{\theta_{ij}} J(w) = (q_{ij} - q_{ij+1})\psi(\xi^i(t), t) b_i(\xi^i(t), t)|_{t=\theta_{ij}} |\Delta\theta_{ij}| +$$
$$+2\alpha_2 (v(\theta_{ij}) - v^0(\theta_{ij}))(q_{ij} - q_{ij+1})|\Delta\theta_{ij}| + o(\|\Delta v(t)\|).$$

Dividing both sides into $\Delta\theta_{ij}$, and proceeding to the limit when $\Delta\theta_{ij} \to 0$, we get:

$$\frac{\partial J(w)}{\partial \theta_{ij}} = (q_{ij} - q_{ij+1})\psi(\xi^i(\theta_{ij}), \theta_{ij}) b_i(\xi^i(\theta_{ij}), \theta_{ij}) + 2\alpha_2 (v(\theta_{ij}) - v^0(\theta_{ij}))(q_{ij} - q_{ij+1}),$$
$$j = 1, ..., m_i - 1, i = 1, ..., L. \tag{38}$$

Similarly it can be shown that the formula for the gradient of the functional with respect to the parameter θ_{ij} coincides with (38) in case $\Delta\theta_{ij} < 0$.

3.1.3. *Sources' controlled powers are from the class of Heaviside functions.* Use formula (24) for the functional's increment to obtain the formulas for the components of the gradient of the functional $\frac{dJ(w)}{dq_i}$, $i = 1, ..., L$ in problem (1)-(4), (11)-(13), when controls are from the class of Heaviside functions. When we increment the argument q_i of the vector w from (12) by Δq_i, i.e.

$$\Delta w = (\Delta_i q, 0) \in R^{2L}, \Delta_i q = (0, ..., \Delta q_i, ..., 0) \in R^L,$$

the functional's increment is of the following form

$$\Delta_{q_i} J(w) = \int\limits_0^T \int\limits_\Omega \psi(x, t) b_i(x, t) \Delta q_i \chi(t - \theta_i) \delta(x - \xi^i(t)) dx dt +$$
$$+2\alpha_2 \int\limits_{\theta_i}^T (v_i(t) - v_i^0(t)) \Delta q_i dt + o(\|\Delta v(t)\|) = \int\limits_{\theta_i}^T \psi(\xi^i(t), t) b_i(\xi^i(t), t) \Delta q_i dt +$$
$$+2\alpha_2 \int\limits_{\theta_i}^T (v_i(t) - v_i^0(t)) \Delta q_i dt + o(\|\Delta v(t)\|).$$

Dividing both sides of the above expression into Δq_i, proceeding to the limit when $\Delta q_i \to 0$, and taking into account that $o(||\Delta v(t)||)/\Delta q_i \to 0$, we obtain

$$\frac{dJ(w)}{dq_i} = \int\limits_{\theta_i}^T \psi(\xi^i(t), t) b_i(\xi^i(t), t) dt + 2\alpha_2 \int\limits_{\theta_i}^T (v_i(t) - v_i^0(t)) dt, i = 1, ..., L. \tag{39}$$

To obtain formulas for the derivatives $\frac{dJ(w)}{d\theta_i}$, $i = 1, ..., L$, increment the argument θ_i of the vector (12) by $\Delta\theta_i$, i.e.

$$\Delta w = (0, \Delta_i \theta) \in R^{2L}, \Delta_i \theta = (0, ..., \Delta\theta_i, ..., 0) \in R^L.$$

First, assume that $\Delta\theta_i > 0$, and Heaviside function obtains an increment in the following form

$$\Delta\chi(t-\theta_i) = \chi(t-(\theta_i+\Delta\theta_i)) - \chi(t-\theta_i) = \begin{cases} 0, & t \notin [\theta_i, \theta_i+\Delta\theta_i], \\ -1, & t \in [\theta_i, \theta_i+\Delta\theta_i], \end{cases}$$

Then the functional obtains the following increment

$$\Delta_{\theta_i} J(w) = \int_0^T \int_\Omega \psi(x,t) b_i(x,t) q_i \delta(x-\xi^i(t)) \Delta\chi(t-\theta_i) dxdt -$$
$$-2\alpha_2 q_i \int_{\theta_i}^{\theta_i+\Delta\theta_i} v_i(t) - v_i^0(t))dt + o(\|\Delta v(t)\|) = -q_i \int_{\theta_i}^{\theta_i+\Delta\theta_i} \psi(\xi^i(t),t) b_i(\xi^i(t),t) dxdt -$$
$$-2\alpha_2 q_i \int_{\theta_i}^{\theta_i+\Delta\theta_i} v_i(t) - v_i^0(t))dt + o(\|\Delta v(t)\|).$$

The increment $\Delta\chi(t-\theta_i)$ of Heaviside function obtains the following form for $\Delta\theta_i < 0$

$$\Delta\chi(t-\theta_i) = \begin{cases} 0, & t \notin [\theta_i - |\Delta\theta_i|,\, \theta_i], \\ 1, & t \in [\theta_i - |\Delta\theta_i|,\, \theta_i]. \end{cases}$$

Then we have the following formula for the functional's increment:

$$\Delta_{\theta_i} J(w) = q_i \int_{\theta_i - |\Delta\theta_i|}^{\theta_i} \psi(\xi^i(t),t) b_i(\xi^i(t),t) dxdt + 2\alpha_2 q_i \int_{\theta_i - |\Delta\theta_i|}^{\theta_i} (v_i(t) - v_i^0(t))dt + o(\|\Delta v(t)\|).$$

The functional's increment will be of the following form, taking the average theorem into account for both $\Delta\theta_i > 0$ and $\Delta\theta_i < 0$

$$\Delta_{\theta_i} J(w) = \mp q_i \psi(\xi^i(\theta_i),\theta_i) b_i(\xi^i(\theta_i),\theta_i) |\Delta\theta_i| \mp 2\alpha_2 (v_i(\theta_i) - v_i^0(\theta_i)) q_i |\Delta\theta_i| + o(\|\Delta v(t)\|),$$

where the signs "+" and "–" correspond to the cases $\Delta\theta_i > 0$ and $\Delta\theta_i < 0$, respectively. Dividing both parts of the above expression into $\Delta\theta_i$, and proceeding to the limit when $\Delta\theta_i \to 0$, regardless of the sign of $\Delta\theta_i$, we obtain:

$$\frac{dJ(w)}{d\theta_i} = -q_i \psi(\xi^i(\theta_i),\theta_i) b_i(\xi^i(\theta_i),\theta_i) - 2\alpha_2 (v_i(\theta_i) - v_i^0(\theta_i)) q_i,\, i = 1, ..., L, \tag{40}$$

Therefore the components of the gradient of the functional in problem (1)-(4), (11)-(13) are determined by formulas (39), (40) in the space of the control parameters $(q,\theta) \in R^{2L}$.

3.2. **The formulas for the components of the gradient of the functional with respect to the sources' placement.** Here we obtain formulas for the gradient of the functional with respect to the control actions on the trajectory of the source's motion, which is determined by system of differential equations (5).

We take any two admissible controls for this purpose: $w = (v(t), s(t))$ and $w + \Delta w = (v(t), s(t) + \Delta_i s(t))$, $\Delta_i s(t) = (0, .., \Delta s_i(t), .., 0)$.

Then the increment of the trajectory of the $i-$th source's motion satisfy the following Cauchy problem:

$$\begin{aligned} &\Delta\dot{\xi}^i(t) = f^i(\xi^i(t) + \Delta\xi^i(t), s_i(t) + \Delta s_i(t), t) - f^i(\xi^i(t), s_i(t), t),\ t \in (0,T], \\ &\Delta\xi^i(0) = 0. \end{aligned} \tag{41}$$

Let $u(x,t;w)$, $u(x,t;w+\Delta w)$ be the solutions to boundary problem (2)-(4) for these controls and the corresponding trajectories of the sources' motion. Introduce the notation

$$\Delta u(x,t) = u(x,t;w+\Delta w) - u(x,t;w).$$

From (2)-(5), it follows that $\Delta u(x,t)$ is a generalized solution to the boundary problem

$$\Delta u_t(x,t) = div\,(\sigma(x) grad\,\Delta u(x,t)) + \sum_{i=1}^{L} v_i(t) b_i(x,t) \Delta_{\xi^i(t)} \delta(x - \xi^i(t)), \tag{42}$$

$$x \in \Omega \subset R^n, 0 < t \le T,$$

$$\Delta u(x,0) = 0, x \in \Omega, \tag{43}$$

$$\Delta u(x,t)|_{x\in\Gamma^1} = 0, \sigma(x)\frac{\partial \Delta u(x,t)}{\partial n}|_{x\in\Gamma^2} = 0, 0 < t \le T. \tag{44}$$

Then the increment of functional (1) can be written as

$$\Delta J(w) = J(w+\Delta w) - J(w) = 2\alpha_1 \int_\Omega [u(x,T;w) - U(x)]\Delta u(x,T)dx +$$

$$+\alpha_1 \int_\Omega |\Delta u(x,T)|^2 dx + \alpha_3(\left\|s(t) + \Delta s(t) - s^0(t)\right\|^2 - \left\|s(t) - s^0(t)\right\|^2).$$

Let $\psi(x,t)$ be a solution to conjugate problem (20)-(22). Carrying out similar computations in case of the optimization of impulsive powers, we have the following formulas for the functional's increment

$$\begin{gathered}\Delta_{\xi^i(t)} J(w) = \int_0^T \int_\Omega \psi(x,t) b_i(x,t) v_i(t) \Delta_{\xi^i(t)} \delta_\varepsilon(x - \xi^i(t)) dx dt + \\ +2\alpha_3 \int_0^T (s_i(t) - s_i^0(t))\Delta s_i(t) dt + o\left(\|\Delta s(t)\|\right).\end{gathered} \tag{45}$$

Carrying out similar computations in case of the optimization of the points of times of the sources' impulsive actions, from (45), we have

$$\Delta_{\xi^i(t)} J(w) = \int_0^T v_i(t) (\psi(\xi^i(t),t) b_i(\xi^i(t),t))'_{\xi^i(t)} \Delta\xi^i(t) dt + o(\left\|\Delta\xi^i(t)\right\|) +$$

$$+2\alpha_3 \int_0^T (s_i(t) - s_i^0(t))\Delta s_i(t) dt + o\left(\|\Delta s(t)\|\right).$$

Introduce an analogue of Hamilton-Pontryagin's function [9] in the following form

$$H(\xi(t),\omega(t),s(t),t) = \sum_{i=1}^{L} \omega^i(t) f^i(\xi^i(t), s_i(t), t) + \sum_{i=1}^{L} v_i(t)\psi(\xi^i(t),t) b_i(\xi^i(t),t).$$

where $\omega^i(t)$ is the solution to the following conjugate problem

$$\dot{\omega}^i(t) = -\frac{\partial H(\xi(t),\omega(t),s(t),t)}{\partial \xi^i(t)},\ t \in (0,T],\ \omega^i(T) = 0. \tag{46}$$

It is evident that

$$\omega^i(T)\Delta\xi^i(T)-\omega^i(0)\Delta\xi^i(0)=\int_0^T \dot{\omega}^i(t)\Delta\xi^i(t)dt+\int_0^T \omega^i(t)\Delta\dot{\xi}^i(t)dt. \tag{47}$$

From (41), (46), (47), it follows that

$$\int_0^T \dot{\omega}^i(t)\Delta\xi^i(t)dt=-\int_0^T \omega^i(t)\Delta\dot{\xi}^i(t)dt,$$

i.e.

$$-\int_0^T \frac{\partial H(\xi(t),\omega(t),s(t),t)}{\partial \xi^i(t)}\Delta\xi^i(t)dt=-\int_0^T \omega^i(t)\frac{\partial f^i(\xi^i(t),\omega^i(t),s_i(t),t)}{\partial\xi^i(t)}\Delta\xi^i(t)dt+$$
$$-\int_0^T \omega^i(t)\frac{\partial f^i(\xi^i(t),\omega^i(t),s_i(t)+\Delta s_i(t),t)}{\partial\xi^i(t)}\Delta\xi^i(t)dt-\int_0^T \omega^i(t)\frac{\partial f^i(\xi^i(t),\omega^i(t),s_i(t),t)}{\partial s_i(t)}\Delta s_i(t)dt=$$
$$=-\int_0^T \omega^i(t)\frac{\partial f^i(\xi^i(t),\omega^i(t),s_i(t),t)}{\partial\xi^i(t)}\Delta\xi^i(t)dt-\int_0^T \omega^i(t)\frac{\partial f^i(\xi^i(t),\omega^i(t),s_i(t),t)}{\partial s_i(t)}\Delta s_i(t)dt+\eta,$$

where $\eta=\eta_1+\eta_2+\eta_3+\eta_4$, $\eta_1=o(\|\Delta\xi^i(t)\|)$, $\eta_2=\int_0^T o_1(\|\Delta\xi(t)\|)$, $\eta_3=-\int_0^T \frac{\partial \Delta_s H(\xi(t),\omega(t),s(t),t)}{\partial\xi^i(t)}$ $\Delta\xi^i(t)dt$, $\eta_4=o(\|\Delta s(t)\|)$,
where $\Delta_s H(\xi(t),\omega(t),s(t),t)=H(\xi(t),\omega(t),s(t)+\Delta s(t),t)-H(\xi(t),\omega(t),s(t),t)$.
Consequently,

$$\int_0^T v_i(t)(\psi(\xi^i(t),t)b_i(\xi^i(t),t))'_{\xi^i(t)}\Delta\xi^i(t)dt=\int_0^T \omega^i(t)\frac{\partial f^i(\xi^i(t),\omega^i(t),s_i(t),t)}{\partial s_i(t)}\Delta s_i(t)dt+\eta.$$

Taking into account, that

$$\begin{aligned}&\Delta_{s_i(t)}J(w)=J(v,\xi(s_i(t)+\Delta s_i(t)))-J(v,\xi(s_i(t))=J(v,\xi(s_i(t))+\Delta\xi(s_i(t))-\\&-J(v,\xi(s_i(t)))=\int_0^T \frac{\partial H(\xi(t),\omega(t),s(t),t)}{\partial s_i(t)}\Delta s_i(t)dt+\\&+2\alpha_3\int_0^T (s_i(t)-s_i^0(t))\Delta s_i(t)dt+\eta,\end{aligned} \tag{48}$$

we have

$$grad_{s_i(t)}J(w)=\frac{\partial H(\xi(t),\omega(t),s(t),t)}{\partial s_i(t)}+2\alpha_3(s_i(t)-s_i^0(t)). \tag{49}$$

Particularly, if the sources are motionless, but the coordinates of their placement are optimized, then the formulas for the gradient of the functional

$$J(w)=\alpha_1\int_\Omega [u(x,T;w)-U(x)]^2dx+\alpha_2\left\|v(t)-v^0(t)\right\|_{L_2}^2+\alpha_3\sum_{i=1}^L \left\|\xi^i-\xi^{i0}\right\|_{R^n}^2$$

with respect to the sources' coordinates are obviously of the following form

$$\frac{dJ(w)}{d\xi_j^i} = \int_0^T v_i(t)(\psi(x,t)b_i(x,t))'_{x_j}|_{x=\xi^i}dt + 2\alpha_3(\xi_j^i - \xi_j^{i0}), j = 1, ..., n, i = 1, ...L. \tag{50}$$

3.2.1. *Control actions on the sources' motion are from the class of impulsive functions.* Let the sources' motion, which is described by system of differential equations (5), be realized by impulsive control. We increment the argument s_{ij} of the vector (14) by Δs_{ij}, i.e. the functional's increment obtained at the expense of the increment $\Delta w = (\Delta_{ij}s, 0) \in R^{2LM}$, $\Delta_{ij}s = (0, ..., \Delta s_{ij}, ..., 0) \in R^{ML}$, can be written as follows

$$\Delta_{s_{ij}} J(w) = \int_0^T \frac{\partial H(\xi(t), \omega(t), s(t), t)}{\partial s_i(t)} \Delta s_{ij}\delta(t - \tau_{ij})dt + 2\alpha_3 \int_0^T (s_i(t) - s_i^0(t)) \times$$
$$\times \Delta s_{ij}\delta(t - \tau_{ij})dt + \eta.$$

Dividing both sides of the above expression into Δs_{ij}, and proceeding to the limit when $\Delta s_{ij} \to 0$, we have the required formulas for the components of the gradient of the functional

$$\frac{dJ(w)}{ds_{ij}} = \frac{\partial H(\xi(\tau_{ij}), \omega(\tau_{ij}), s(\tau_{ij}), \tau_{ij})}{\partial s_i(\tau_{ij})} s_{ij} + 2\alpha_3(s_i(\tau_{ij}) - s_i^0(\tau_{ij})), j = 1, ..., m_i, i = 1, ..., L.$$

3.2.2. *Control actions on the sources' motion are from the class of piecewise constant functions.* Let the sources' motion, which is described by system of differential equations (5), is realized by controls from the class of piecewise constant functions. We choose the increment $\Delta s_i(t)$ in the following form:

$$\Delta s_i(t) = \begin{cases} 0, & 0 < t < \tau_{ij-1},\ \tau_{ij} < t < T, \\ \Delta s_{ij} = const, & \tau_{ij-1} \le t < \tau_{ij},\ j = 1, ..., m_i,\ i = 1, ..., L. \end{cases}$$

Taking this increment into account in (48), we have the formulas for the components of the gradient of the functional in the following form

$$\frac{dJ(w)}{ds_{ij}} = \int_{\tau_{ij-1}}^{\tau_{ij}} \frac{\partial H(\xi(t), \omega(t), s(t), t)}{\partial s_i(t)} dt + 2\alpha_3 \int_{\tau_{i-1}}^{\tau_{ij}} (s_i(t) - s_i^0(t))dt, j = 1, ..., m_i, i = 1, ..., L.$$

3.2.3. *Control actions on the sources' motion are from the class of Heaviside functions.*

Let the sources' motion, which is described by system of differential equations (5), is realized by relay controls from the class of Heaviside functions. The functional's increment obtained at the expense of the increment Δw, $\Delta w = (\Delta_i s, 0) \in R^{2L}$, $\Delta_i s = (0, ..., \Delta s_i, ..., 0) \in R^L$ (increment of the argument s_i of vector (16)), can be written as follows

$$\Delta_{s_i} J(w) = \int_0^T \frac{\partial H(\xi(t), \omega(t), s(t), t)}{\partial s_i(t)} \Delta s_i \chi(t - \tau_i)dt + 2\alpha_3 \int_0^T (s_i(t) - s_i^0(t)) \times$$
$$\times \Delta s_i \chi(t - \tau_i)dt + \eta.$$

Dividing both parts of the above expression into Δs_i, and proceeding to the limit when $\Delta s_i \to 0$, we have the following formulas for the components of the gradient of the functional in problem (1)-(7)

$$\frac{dJ(w)}{ds_i} = \int_{\tau_i}^{T} \frac{\partial H(\xi(t), \omega(t), s(t), t)}{\partial s_i(t)} dt + 2\alpha_3 \int_{\tau_i}^{T} (s_i(t) - s_i^0(t))dt, i = 1, ..., L.$$

4. The results of numerical experiments

Problem 1. Consider a problem of the heating of a stick by lumped sources with impulsive action, when $L = 1$, i.e. we can apply only one impulsive action on the process.

$$u_t = u_{xx} + (x + t)q\delta(x - \xi)\delta(t - \theta), 0 < x < 1, \ 0 < t \le 1,$$

$$u(x, 0) = e^x, 0 \le x \le 1, u(0, t) = t + 1, \ \ u(1, t) = e^{t+1}, 0 < t \le 1,$$

$$0 \le \xi \le 1, \ \ 0 \le \theta \le 1, \ \ 0 < q \le 10,$$

$$J(w) = \int_0^1 [u(x, 1) - 4]^2 dx + 0,1(q - 3)^2 + 0,1(\xi - 0,5)^2 + 0,1(\theta - 0,3)^2 \to \min .$$

Thus the optimized parameters are the power, action time and the coordinate of the source's impulsive action $w = (q, \theta, \xi)$.

The exact value of the optimized vector is unknown. The problem is solved numerically by using the formulas obtained above.

The results of the numerical experiments by using the interfaced gradient projection method are given in table 1 with various initial values of the control parameters $w^0 = \left(q^0, \theta^0, \xi^0\right)$, with the precision of the optimization $\varepsilon = 0,001$. Approximation of the boundary problem is made by using the implicit scheme of grid method with error $O(h_x^2 + h_t)$ including boundary conditions, where $h_x = 0,01$ and $h_t = 0,01$ are grid steps on the variables x and t, respectively.

Problem 2.

$$u_t = u_{xx} + (x^2 + t^2) \sum_{i=1}^{2} q_i \delta(x - \xi_i)\delta(t - \theta_i), 0 < x < 1, \ 0 < t \le 1,$$

$$u(x, 0) = e^x, 0 \le x \le 1,$$

$$u(0, t) = t + 1, \ \ u(1, t) = e^{t+1}, 0 < t \le 1,$$

$$0 \le \xi_i \le 1, \ \ 0 \le \theta_i \le 1, 0 < q_i \le 10, \ i = 1, 2,$$

$$J(v) = \int_0^1 [u(x, 1) - 4]^2 dx + 0,1((q_1 - 3)^2 + (q_2 - 4)^2 +$$
$$+ 0,1((\xi_1 - 0,5)^2 + (\xi_2 - 0,8)^2) + 0,1((\theta_1 - 0,3)^2 + (\theta_2 - 0,5)^2) \to \min .$$

$L = 2$ in this problem. The exact value of the optimized vector $w = (q, \xi, \theta)$, $q, \xi, \theta \in R^2$ is unknown. The problem is solved numerically by using the formulas obtained above. The results of numerical experiments by using the interfaced gradient projection method are given in table 2 with various initial values of the control vector $w^0 = \left(q^0, \theta^0, \xi^0\right)$, with the precision of the optimization $\varepsilon = 0,001$. Approximation of the boundary problem is made similarly to the previous problem.

Note that similar researches can be carried out in processes described by other types of partial differential equations.

Table 1. The numerical results of the problem 1.

	(q^0, ξ^0, θ^0)	(q^*, ξ^*, θ^*)	J^0	J^*	Number of iterations
1	(3;0,6;0,1)	(2,995;0,479;0,219)	2,973	2,568	9
2	(6;0,2;0,2)	(2,998;0,490;0,229)	3,477	2,568	4
3	(1;0,2;0,4)	(3,00;0,494;0,219)	2,978	2,568	3

Table 2. The numerical results of the problem 2.

	(q^0, ξ^0, θ^0)	(q^*, ξ^*, θ^*)	J^0	J^*	Number of iterations
1	(1;3;0,2; 0,2;0,4;0,3)	(3,026; 4,008;0,491; 0,860;0,258;0,299)	3,1177	2,5735	11
2	(2;6;0,4;0.6;0,74;0,6)	(3,00; 3,985;0,486; 0,888;0,260;0,369)	3,3466	2,5727	12
3	(2;4;0,3;0.5;0,2;0,3)	(2,998; 3,99;0,488; 0,911;0,259;0,359)	2,68673	2,5729	36
4	(1;2;0.5;0.5;0,2;0,1)	(2,998; 3,996;0,488; 0,910;0,259;0,367)	3,3933	2,5728	15

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