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FIBONACCI LENGTH AND SPECIAL AUTOMORPHISMS OF FINITE $(\ell, m|n, k)$-GROUPS

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Abstract. We study here the Fibonacci length of finitely presented and parametric groups $\langle a, b \mid a^\ell = b^m, (ab)^n = (ab^{-1})^k \rangle$ for positive integers ℓ, m, n and k. They are indeed extensions of $(\ell, m|n, k)$-groups of M. Edjvet and R.M. Thomas considered for their finiteness property in 1997. We prove that there are some subclasses of these groups which are non-isomorphc groups of the same Fibonacci length. More interesting result is that, these lengths are independent of one of the involved parameters of the groups, and also the lengths involve the Wall number $\kappa(n)$. Moreover, the Fibonacci lengths of two homomorphic images of the groups have been calculated and compared with those of the groups.

Keywords: groups, Fibonacci lengths, special automorphisms.

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1. Introduction

The 4-parameter groups

$$G = (\ell, m|n, k) = \langle a, b \mid a^\ell = b^m = (ab)^n = (ab^{-1})^k = 1 \rangle$$

were studied in 1997 by M. Edjvet and R.M. Thomas (see [10]) where, $\ell, m, n, k \geq 1$, $\ell \leq m$ and $n \leq k$. The study of these groups is a continuation of the study of the Coxeter groups of [5, 6] and the groups

$$\overline{G} = \langle a, b \mid a^\ell = b^m = 1, (ab)^n = (ab^{-1})^k \rangle$$

and

$$\overline{\overline{G}} = \langle a, b \mid a^\ell = b^m, (ab)^n = (ab^{-1})^k \rangle$$

considered by H. Doostie, R. Gholamie and R.M. Thomas in 2003 (see [9]). It is shown in [9] that the group G is finite only in the following cases:

$(2, 2\|n, k)$,	$n \geq 2$;	$(\ell, m\|2, 2)$,	$\ell \geq 4$;
$(3, 3\|3, k)$,	$k \geq 3$;	$(7, 8\|2, 3)$;	
$(3, 3\|4, 4,)$;		$(3, 4\|3, 4)$;	
$(3, 3\|4, 5)$;		$(3, 4\|3, 5)$;	
$(3, 5\|3, 4)$;		$(4, m\|2, 3)$,	$m = 6, 7, 8, 9$;
$(4, 5\|2, k)$,	$k = 3, 4, 5$;	$(5, 5\|2, 4)$;	
$(6, 7\|2, 3)$;		$(7, 7\|2, 3)$;	

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$(4,4\|2,k)$,	$k \geq 3$;	$(5,m\|2,3)$,	$m \geq 3$;
$(2,3\|n,k)$,	$3 \leq gcd(n,k) \leq 5$;	$(2,3\|n,k)$,	$3 \leq gcd(n,k) \leq 5$;
$(2,m\|n,k)$,	$m \geq 3,\ gcd(n,k) \leq 2$;	$(2,m\|n,k)$,	$m = 4,5,\ gcd(n,k) = 3$;
$(5,m\|2,k)$,	$gcd(m,k) \leq 5$;	$(3,m\|3,3)$,	$m \geq 4$.

We consider all of the finite cases of G, $\overline{G}$ and $\overline{\overline{G}}$ and study the Fibonacci length of them. This computation compares the Fibonacci length of the extensions of G and gives us explicit formulas for lengths which are in certain cases independent of at least one of the parameters of the groups.

The periodic sequences of elements of finite algebraic structures have been studied by many authors, one may see [2, 5, 3, 4, 10, 9], for examples. Following the notations of the article [4] where, for a 2-generated non-abelian finite group $G = \langle a, b\rangle$,

$$x_1 = a,\ x_2 = b,\ x_i = x_{i-2}x_{i-1}, \quad i \geq 3$$

is called the Fibonacci sequence of G depending the generating set $\{a, b\}$. The least integer k (denoted by $LEN(G)$) such that $x_{k+1} = x_1$ and $x_{k+2} = x_2$ is called the Fibonacci Length of G, and the least integer m (denoted by $BLEN(G)$) such that $|x_{m+1}| = |x_1|$ and $|x_{m+2}| = |x_2|$, is called the basic Fibonacci Length of G, where the $|x|$ denotes the order of the element x in the group G. Note, it is proved that $BLEN$ divides LEN and the map $\theta : G \longrightarrow G$ given by $a \mapsto x_{m+1}$ and $b \mapsto x_{m+2}$ is an automorphism of G with order $LEN/BLEN$ (for more details one may refer to [4]). We call this θ, the *special* automorphism of G. In 1990 Campbell, Doostie and Robertson [4] gained to compute the length of the groups D_{2n} and Q_{2^n}, and the simple groups of order less than 10^5. The Fibonacci length and the basic Fibonacci length of the groups $Aut(D_{2n})$ and $Aut(Q_{2^n})$ have been computed in 2000 by Doostie and Campbell (see [7]).

The Fibonacci sequence $\{f_n\}_{n=0}^{\infty}$ of numbers defined by $f_n = f_{n-2} + f_{n-1}$ for $n \geq 0$, and we seed the sequence with $f_0 = 0$ and $f_1 = 1$. Modulo some integer $n \geq 2$, it must ultimately become periodic as there are only n^2 different pairs of residues modulo n. Further, throughout this paper, we use $\kappa(n)$ to denote the fundamental period of the Fibonacci sequence modulo n, and call it the Wall number of n (see [12]).

We attempt here to calculate the Fibonacci Lengths of certain remained cases of $\overline{G}$ and $\overline{\overline{G}}$ of the article [9].

2. The groups $(2,2|n,k),\ (n \geq 2,\ k \geq 2,\ n \leq k)$

As a result of Proposition 2.1 of [9] we get the finite cases as follows:

$$\begin{cases} \text{if} \quad d = gcd(n,k) & \text{then} \quad G \cong D_{2d}, \text{ the dihedral groups of order } 2d, \\ \text{if} \quad n \neq k & \text{then} \quad \overline{G} \cong D_{2\|n-k\|}, \\ \text{if} \quad n \neq k & \text{then} \quad \overline{\overline{G}}, \text{ is finite of order } 4n\|n-k\|. \end{cases}$$

As a quick result of [4] we deduce that $LEN_{\{a,b\}}(G) = LEN_{\{a,b\}}\overline{G} = 6,\ BELN_{\{a,b\}}(G) = 3$ and $\theta : G \to G$ will be the special automorphism and then

$$\theta : \begin{cases} a & \mapsto & bab, \\ b & \mapsto & b. \end{cases}$$

To calculate the Fibonacci length of $\overline{\overline{G}}$ we consider the Fibonacci sequence $\{f_i\}$ of numbers as

$$f_1 = f_2 = 1,\ f_{i+1} = f_i + f_{i-1}, \quad i \geq 2.$$

and define the Fibonacci sequence of elements of $\overline{\overline{G}}$ as usual by:

$$x_1 = a,\ x_2 = b,\ x_i = x_{i-2}x_{i-1}, \quad i \geq 3.$$

Then, we have:

Lemma 2.1. *For every integers n and k where $n \neq k$ and $n, k \geq 2$, every element of the sequence $\{x_i\}$ may be presented by*

$$x_i = \begin{cases} a^{f_i}, & if\ i \equiv 1 \pmod 6, \\ a^{-1+f_i}b, & if\ i \equiv 2, 3, -1 \pmod 6, \\ a^{-3+f_i}bab, & if\ i \equiv -2 \pmod 6, \\ a^{-2+f_i}ba, & if\ i \equiv 0 \pmod 6. \end{cases}$$

Proof. For $i \leq 6$ we get the result at once. then using an induction method (by considring six cases) we may get the result. For example let $i \equiv 1\ (mod\ 6)$ then, $i - 1 \equiv 0\ (mod\ 6)$ and $i - 2 \equiv -1\ (mod\ 6)$. So,

$$\begin{aligned} x_i &= x_{i-2}x_{i-1} \\ &= a^{-1+f_{i-2}} \cdot b \cdot a^{-2+f_{i-1}} \cdot ba && \text{(by induction hypothesis)} \\ &= a^{-1+f_{i-2}} \cdot a^{-2+f_{i-1}} \cdot b^2a && \text{(for, } a^2 = b^2 \text{ and } f_{i-1} \text{ is even in this case)} \\ &= a^{-3+f_i} \cdot a^2a && \text{(for, } a^2 = b^2) \\ &= a^{f_i}. \end{aligned}$$

as required. □

Proposition 2.1. *For every integers n and k where $2 \leq n \leq k$, the Fibonacci length of the group $\overline{\overline{G}}$ is independent of k. $t = LEN_{\{a,b\}}\overline{\overline{G}}$ if and only if $t \equiv 0\ (mod\ 6)$ and $f_{t+3} \equiv 2\ (mod\ 4n)$.*

Proof. As a result of the proof of Proposition 2.1 of [9] we conclude the validity of the relation $a^{4n} = 1$ in the group $\overline{\overline{G}}$. So, $\overline{\overline{G}}$ has a presentation isomorphic to

$$\langle a, b \mid a^{4n} = 1,\ a^2 = b^2,\ (ab)^n = (ab^{-1})^k \rangle.$$

Now, let $t = LEN_{\{a,b\}}\overline{\overline{G}}$. Then, t is the least positive integer such that the equations $x_{t+1} = x_1$ and $x_{t+2} = x_2$ hold in $\overline{\overline{G}}$. Equivalenty we get:

$$t + 1 \equiv 1\ (mod\ 6),\ a^{f_{t+1}} = a,\ a^{-1+f_{t+2}}.b = b$$

by considering the Lemma 2.1 and its possible cases. These equations yield in turn the numerical equations

$$\begin{cases} f_{t+1} \equiv 1 \pmod{4n} \\ f_{t+2} \equiv 1 \pmod{4n} \\ t \equiv 0 \pmod 6, \end{cases}$$

by considering the relation $a^{4n} = 1$ of $\overline{\overline{G}}$. Consequently, we get $f_{t+3} \equiv 2\ (mod\ 4n)$ and $t \equiv 0\ (\ mod\ 6)$, as required. □

Proposition 2.2. *For every integers n and k where $n \leq k$, $LEN_{\{a,b\}}\overline{\overline{G}} = \kappa(2^2 \cdot n)$. So, the Fibonacci length of the group $\overline{\overline{G}}$ is independent of k.*

Proof. It is similar to the proof of 2.1. □

3. The groups $(\ell, m|2, 2),\ (\ell \geq 4,\ m \geq 2)$

Let

$$\overline{\overline{G}} = \langle a, b \mid a^{\ell} = b^{m},\ (ab)^2 = (ab^{-1})^2 \rangle.$$

By the Lemma 3.1 of [9], $|\overline{\overline{G}}| = 4\ell$ and its proof shows that the relation $a^{4m} = 1$ holds in $\overline{\overline{G}}$. Similar to the last section, every element of the Fibonacci sequence of elements of $\overline{\overline{G}}$ may be presented as follows

$$x_1 = a, x_2 = b,$$

$$x_k = \begin{cases} a^{f_{k-2}}, & \text{if } \ k \equiv 1, -2 \quad (mod\ 6), \\ a^{f_{k-2}} \cdot b, & \text{if } \ k \equiv 2 \text{ or } 3 \quad (mod\ 6), \\ a^{f_{k-2}} \cdot b^{-1}, & \text{if } \ k \equiv 0 \text{ or } -1 \quad (mod\ 6), \end{cases}$$

for every $k \geq 3$. The proof is easy by considering different cases for k and by using an induction method. Using this feat we get the following result:

Proposition 3.1. (i) *For non-abelian cases of G (at least one of ℓ and m is even), $LEN(G) = 6$,*

(ii) *For every even value of ℓ and odd values of m, $t = LEN_{\{a,b\}}\overline{G}$ if and only if $f_{t-2} \equiv 1 (mod\ 4\ell)$ and $f_{t-1} \equiv 0\ (mod\ 4\ell)$,*

(iii) *For every integers ℓ and m where $\ell \geq 4$, the Fibonacci length of the group $\overline{\overline{G}}$ is independent of m, and $LEN_{\{a,b\}}\overline{\overline{G}} = \kappa(2^2 \cdot \ell)$.*

Proof. (i) It is sufficient to consider the Fibonacci sequences of elements $x_1 = a,\ x_2 = b,\ x_3 = ab,\ x_4 = bab = a^{-1},\ x_5 = aba^{-1},\ x_6 = ba^{-1},\ x_7 = aba^{-1}ba^{-1} = a,\ x_8 = b$. So $LEN(G) = 6$ for all values of ℓ and m whenever at least one of them is even. *(ii)* In $\overline{\overline{G}}$ and $\overline{G}$ the relation $(ab)^2 = (ab^{-1})^2$ yields $aba^{-1} = b^{-1}$ (for m odd) and then we get $a^t ba^{-t} = b^{\pm 1}$ if t is even or odd, respectively. Then every element of the Fibonacci sequence x_i of $\overline{\overline{G}}$ and $\overline{G}$ will be represented as:

$$x_k = \begin{cases} a^{f_{k-3}}, & \text{if } \ k \equiv 1, 4 \quad (mod\ 6), \\ a^{f_{k-3}} \cdot b, & \text{if } \ k \equiv 2 \text{ or } 3 \quad (mod\ 6), \\ a^{f_{k-3}} \cdot b^{-1}, & \text{if } \ k \equiv 0 \text{ or } -1 \quad (mod\ 6), \end{cases}$$

where $k \geq 7$, and $x_1 = a,\ x_2 = b,\ x_3 = ab,\ x_4 = a,\ x_5 = a^2b^{-1},\ x_6 = a^3b^{-1}$. The proof is straight forward and follows by induction on k. Let $t = LEN(\overline{G})$. Then $x_{t+1} = x_1$ and $x_{t+2} = x_2$; i.e. $t \equiv 0\ (mod\ 6)$, $f_{t-2} \equiv 1\ (mod\ \ell)$ and $f_{t-1} \equiv 0\ (mod\ \ell)$ as required. On the other hand, the relation $b^{4m} = 1$ holds in $\overline{\overline{G}}$ (see [9]). So $a^{4\ell} = 1$ holds in $\overline{\overline{G}}$.

(iii) The proof is similar to the proof of *(ii)*. □

4. The groups $(2, 3|n, k),\quad (3 \leq gcd(n, k) \leq 5)$

For the values $gcd(n, k) = 3, 4, 5$ we have the groups

$$G_1 = \langle a, b \mid a^2 = b^3 = (ab)^3 = 1 \rangle,$$

$$G_2 = \langle a, b \mid a^2 = b^3 = (ab)^4 = 1 \rangle,$$

$$G_3 = \langle a, b \mid a^2 = b^3 = (ab)^5 = 1 \rangle,$$

respectively. Then we deduce:

Proposition 4.1. *$LEN(G_1) = 16$, $LEN(G_2) = 18$, $LEN(G_3) = 50$. Moreover, the special automorphisms are given by:*

$$\theta_1 : \begin{cases} a & \mapsto & a, \\ b & \mapsto & ba, \end{cases} \qquad \theta_2 : \begin{cases} a & \mapsto & a, \\ b & \mapsto & aba, \end{cases} \qquad \theta_3 : \begin{cases} a & \mapsto & bab^{-1}, \\ b & \mapsto & bab^{-1}ab^{-1}. \end{cases}$$

Proof. The Fibonacci sequence of elements for G_1 are

$$a, b, ab, bab, ab^{-1}ab, abab, ab^{-1}, b, a, ba, aba,$$

$$b^{-1}a, abab^{-1}a, ab^{-1}a, b^{-1}, ba, a, b,$$

which gives that $LEN(G_1) = 16$. Considering the orders yields $BLEN(G_1) = 8$, and θ_1 comes immediately. For G_2 we do as before and simplify the words. We get:

$$a, b, ab, bab, ab^{-1}ab, ab^{-1}abab, b^{-1}ab^{-1}, ab^{-1}a, ab,$$

$$a, aba, ba, b^{-1}ab^{-1}, b^{-1}aba, bab^{-1}, bab, b^{-1}, ba, a, b.$$

So, $LEN(G_2) = 18$, $BLEN(G_2) = 9$, and θ_2 will be defined as required. To simplify the words of G_3 we have to use some extra relations of G_3. Using $a^2 = b^3 = (ab)^5 = 1$, we will get the identities:

$$(ab^2ab)^5 = 1,$$

$$(ab^2ab^2ab)^3 = 1,$$

$$(ab^2ab^2abab^2ab)^2 = 1,$$

in G_3. Now the long words of the Fibonacci sequence has to be simplified by hand calculations. So, we get:

$$a, b, ab, bab, ab^2ab, babab^2ab, ab^2ab^2abab^2ab, b^{-1}aba,$$

$$bababb^{-1}, bab, bab^{-1}, bab^{-1}, bab^{-1}ab^{-1}, ab^{-1}, b^{-1}aba,$$

$$ababa, b^{-1}abb^{-1}aba, abab^{-1}abab^{-1}a, ab^{-1}ab^{-1}a, abab^{-1}abababb^{-1}a,$$

$$b^{-1}aba, abab^{-1}abababb^{-1}a, b^{-1}aba, abab^{-1}abababb^{-1}ab^{-1}aba, abab^{-1}ab^{-1}abab^{-1}a,$$

$$b^{-1}ab^{-1}, ab^{-1}ab^{-1}a, ab, ab^{-1}a, abab^{-1}a, b^{-1}a, abab^{-1}ab^{-1}a, ab^{-1}ab^{-1}a,$$

$$abab^{-1}abab^{-1}a, abababb^{-1}a, ab^{-1}ab, b^{-1}a, ab^{-1}, b, a, ba, aba, b^{-1}a, abab^{-1}a,$$

$$ab^{-1}a, ababa, ba, b^{-1}ab^{-1}, bab^{-1}ab^{-1}, bab^{-1}, bab, b^{-1}, ba, a, b.$$

Now, by simple calculations we can show that all of the consecutive pairs before the 51st term is not equal to the generators set, i.e. $\{a, b\}$, except the first and second terms. Therefore, $LEN(G_3) = 50$, $BLEN(G_3) = 10$ and

$$\theta_3 : \begin{cases} a & \mapsto & bab^{-1}, \\ b & \mapsto & b(ab^{-1})^2. \end{cases}$$

This completes the proof. □

5. The groups $(2, m|n, k)$, $(m \geq 3,\ gcd(n, k) \leq 2)$

If $gcd(n, k) = 1$ then G is abelian and there is nothing to calculate, and if $gcd(n, k) = 2$, then

$$G = \langle a, b \mid a^2 = b^m = 1,\ (ab)^2 = 1 \rangle \cong D_{2m}$$

So, $LEN = 6$, $BLEN = 3$.

6. The groups $(2, m|n, k),\ (gcd(n, k) = 3,\ m = 4, 5)$

For $m = 4$, G is equal to:

$$G = (2, 4|n, k) = \langle a, b \mid a^2 = b^4 = (ab)^3 \rangle \cong \langle a, c \mid a^2 = c^3 = (ac)^4 = 1 \rangle.$$

So, as we showed in Section 4, $LEN(G) = 18$. For $m = 5$, we have $G \cong \langle a, c \mid a^2 = c^3 = (ac)^5 = 1 \rangle$, and so $LEN(G) = 50$.

7. Special and remaining cases.

In the sixteen finite cases for G, not all of the groups $\overline{G}$ and $\overline{\overline{G}}$ are finite (see [10, 9]). In the following table we have collected the computer results for the orders and Fibonacci lengths (the codes were written in GAP ([11]).

(ℓ, m, n, k)	$\lvert G\rvert$	$\lvert\overline{G}\rvert$	$\lvert\overline{\overline{G}}\rvert$	$LEN(G)$	$LEN(\overline{G})$	$LEN(\overline{\overline{G}})$
$(7, 8, 2, 3)$	10752	∞	∞	256	–	–
$(3, 3, 4, 4)$	168	1008	8064	10	120	120
$(3, 4, 3, 4)$	168	∞	∞	14	–	–
$(3, 3, 4, 5)$	1080	3240	51840	22	88	264
$(3, 4, 3, 5)$	1080	∞	∞	20	–	–
$(3, 5, 3, 4)$	1080	1080	34560	20	20	240
$(4, 6, 2, 3)$	120	3840	∞	30	30	–
$(4, 7, 2, 3)$	168	168	4368	128	128	2688
$(4, 8, 2, 3)$	1152	∞	∞	36	–	–
$(4, 9, 2, 3)$	2448	2448	∞	480	480	–
$(4, 5, 2, 3)$	1	5	15	–	–	–
$(4, 5, 2, 4)$	160	320	4480	30	30	240
$(4, 5, 2, 5)$	360	360	28080	128	128	896
$(5, 5, 2, 4)$	360	3600	43200	80	240	240
$(6, 7, 2, 3)$	1092	1092	∞	392	392	–
$(7, 7, 2, 3)$	1092	7644	∞	30	240	–

The remaining cases for $\overline{G}$ and $\overline{\overline{G}}$ which have not been studied for their finiteness property are as follows:

$\mathbf{LEN}(\overline{\overline{\mathbf{G}}})$:

$(4, 4|2, k)$, $\quad k = 4, 5$;

$(2, 3|n, k)$, $\quad gcd(n, k) = 5$ and $\dfrac{n + k}{5} \equiv \pm 1\ (mod\ 3)$;

$(2, 3|n, k)$ $\quad gcd(n, k) = 3, 4$;

$(2, 4|n, k)$ and $(2, 5|n, k)$, $\quad gcd(n, k) = 3$;

$(3, m|n, k)$; $\quad gcd(m, k) = 1, \ldots, 5$;

$(2, m|n, k)$, $\quad m \geq 3,\ gcd(m, k) = 1, 2$;

$(5, m|2, 3)$, $\quad m \geq 3$.

$$\mathbf{LEN}(\overline{\mathbf{G}}):$$

$$\begin{array}{ll}(3,m|2,k), & gcd(m,k)=1,\ldots,5;\\ (5,m|2,3), & m\geq 3;\\ (2,3|n,k), & gcd(n,k)=5 \text{ and } \dfrac{n+k}{5}\equiv \pm 1\ (mod\ 3);\\ (2,5|n,k), & gcd(n,k)=3.\end{array}$$

8. Conjectures

Using some procedures in GAP ([11]), related to the computations of Fibonacci length, we give the following conjectures on the groups of $(3,3|3,k),\ k\geq 3$:

(i) For every $k\geq 3$, $LEN(G)=\frac{8k}{gcd(k,3)}$ and,

$$LEN(\overline{G})=\begin{cases}24k, & \text{if } k=6+2^{t+1},\\ 8k, & \text{otherwise.}\end{cases}$$

(ii) Let $\overline{\overline{G}}=U_k$. Then, $LEN(U_{2^t})=LEN(U_{3\cdot 2^t})=3\cdot 2^{t+3}$.

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