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ON NORMAL RULED SURFACES OF GENERAL HELICES IN THE SOL SPACE $\mathfrak{Sol}^3$

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Abstract. In this paper, normal ruled surface of general helices in the $\mathfrak{Sol}^3$ are studied. Also, explicit parametric equations of normal ruled surface of general helices in the $\mathfrak{Sol}^3$ are found.

Keywords: general helix, Sol space, normal surface.

AMS Subject Classification: 53A04, 53A10.

1. Introduction

A developable surface, or torse, is a surface with zero Gaussian curvature. So it can be flattened onto a plane without distortion. In elementary differential geometry, it is stated that under the assumption of sufficient differentiability, a developable surface is either a plane, conical surface, cylindrical surface or tangent surface of a curve or a composition of these types. Thus a developable surface is a ruled surface, where all points of the same generator line share a common tangent plane.

Design using free-form developable surfaces plays an important role in the manufacturing industry. Currently most commercial systems can only support converting free-form surfaces into approximate developable surfaces. Direct design using developable surfaces by interpolating descriptive curves is much desired in industry.

In this paper, normal ruled surface of general helices in the $\mathfrak{Sol}^3$ are studied. It ends with the determination of explicit parametric equations of normal ruled surface of general helices in the $\mathfrak{Sol}^3$, apparently new.

2. Riemannian structure of Sol space $\mathfrak{Sol}^3$

Sol space, one of Thurston's eight 3-dimensional geometries, can be viewed as $\mathbb{R}^3$ provided with Riemannian metric

$$g_{\mathfrak{Sol}^3} = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2, \tag{1}$$

where (x, y, z) are the standard coordinates in $\mathbb{R}^3$.

Note that the Sol metric can also be written as:

$$g_{\mathfrak{Sol}^3} = \sum_{i=1}^{3} \omega^i \otimes \omega^i, \tag{2}$$

where

$$\omega^1 = e^z dx, \quad \omega^2 = e^{-z}dy, \quad \omega^3 = dz, \tag{3}$$

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and the orthonormal basis dual to the 1-forms is

$$\mathbf{e}_1 = e^{-z}\frac{\partial}{\partial x}, \quad \mathbf{e}_2 = e^{z}\frac{\partial}{\partial y}, \quad \mathbf{e}_3 = \frac{\partial}{\partial z}. \tag{4}$$

Proposition 1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric $g_{\mathfrak{Sol}^3}$, defined above the following is true:*

$$\nabla = \begin{pmatrix} -\mathbf{e}_3 & 0 & \mathbf{e}_1 \\ 0 & \mathbf{e}_3 & -\mathbf{e}_2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5}$$

where the (i,j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for the basis

$$\{\mathbf{e}_k, k = 1,2,3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

Lie brackets can be easily computed as:

$$[\mathbf{e}_1, \mathbf{e}_2] = 0, \quad [\mathbf{e}_2, \mathbf{e}_3] = -\mathbf{e}_2, \quad [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_1.$$

The isometry group of $\mathfrak{Sol}^3$ has dimension 3. The connected component of the identity is generated by the following three families of isometries:

$$\begin{aligned} (x,y,z) &\rightarrow (x+c, y, z), \\ (x,y,z) &\rightarrow (x, y+c, z), \\ (x,y,z) &\rightarrow \left(e^{-c}x, e^{c}y, z+c\right). \end{aligned}$$

3. General helices in Sol space $\mathfrak{Sol}^3$

Assume that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{6}$$

where κ is the curvature of γ and τ its torsion and

$$\begin{aligned} &g_{\mathfrak{Sol}^3}(\mathbf{T},\mathbf{T}) = 1, \; g_{\mathfrak{Sol}^3}(\mathbf{N},\mathbf{N}) = 1, \; g_{\mathfrak{Sol}^3}(\mathbf{B},\mathbf{B}) = 1, \\ &g_{\mathfrak{Sol}^3}(\mathbf{T},\mathbf{N}) = g_{\mathfrak{Sol}^3}(\mathbf{T},\mathbf{B}) = g_{\mathfrak{Sol}^3}(\mathbf{N},\mathbf{B}) = 0. \end{aligned} \tag{7}$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, it is possible to write

$$\begin{aligned} \mathbf{T} &= T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3, \\ \mathbf{N} &= N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3, \\ \mathbf{B} &= \mathbf{T}\times\mathbf{N} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3. \end{aligned} \tag{8}$$

Theorem 3.1. *([14]) Let $\gamma : I \longrightarrow \mathfrak{Sol}^3$ be a unit speed non-geodesic general helix. Then, the parametric equations of γ are*

$$\begin{aligned} x(s) &= \tfrac{\sin\mathfrak{P} e^{-\cos\mathfrak{P}s - \mathfrak{C}_3}}{\mathfrak{C}_1^2 + \cos^2\mathfrak{P}}\left[-\cos\mathfrak{P}\cos\left[\mathfrak{C}_1 s + \mathfrak{C}_2\right] + \mathfrak{C}_1\sin\left[\mathfrak{C}_1 s + \mathfrak{C}_2\right]\right] + \mathfrak{C}_4, \\ y(s) &= \tfrac{\sin\mathfrak{P} e^{\cos\mathfrak{P}s + \mathfrak{C}_3}}{\mathfrak{C}_1^2 + \cos^2\mathfrak{P}}\left[-\mathfrak{C}_1\cos\left[\mathfrak{C}_1 s + \mathfrak{C}_2\right] + \cos\mathfrak{P}\sin\left[\mathfrak{C}_1 s + \mathfrak{C}_2\right]\right] + \mathfrak{C}_5, \\ z(s) &= \cos\mathfrak{P}s + \mathfrak{C}_3, \end{aligned} \tag{9}$$

where $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5$ are constants of integration.

An example of a graphic defined by Eq. (9) is illustrated in Fig. 1:

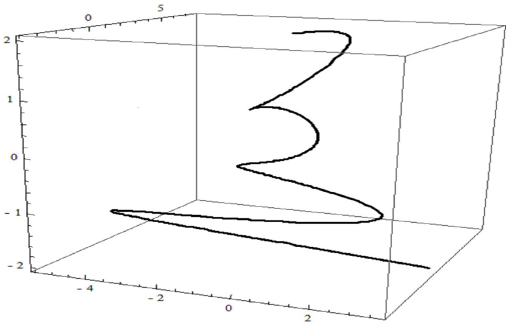

Figure 1.

It results from the above theorem:

Theorem 3.2. *Let* $\gamma : I \longrightarrow \mathfrak{Sol}^3$ *be a unit speed non-geodesic general helix. Then, the equation of* γ *is*

$$\begin{aligned}\gamma(s) = {} & [\tfrac{\sin\mathfrak{P}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\cos\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]+\mathfrak{C}_1\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]+\mathfrak{C}_4 e^{\cos\mathfrak{P}s+\mathfrak{C}_3}]\mathbf{e}_1+ \\ & +[\tfrac{\sin\mathfrak{P}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\mathfrak{C}_1\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]+\mathfrak{C}_5 e^{-\cos\mathfrak{P}s-\mathfrak{C}_3}]\mathbf{e}_2+ \\ & +[\cos\mathfrak{P}s+\mathfrak{C}_3]\mathbf{e}_3,\end{aligned} \tag{10}$$

where $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5$ *are constants of integration.*

4. Normal ruled surfaces of general helices in $\mathfrak{Sol}^3$

The purpose of this section is to study normal ruled surfaces of general helices in $\mathfrak{Sol}^3$.

The normal ruled surface of γ is

$$\mathfrak{L}(s,u) = \gamma(s) + u\mathbf{N}. \tag{11}$$

Theorem 4.1. *Let* $\gamma : I \longrightarrow \mathfrak{Sol}^3$ *is a unit speed non-geodesic general helix in* $\mathfrak{Sol}^3$*. Then, the equation of normal ruled surface of* γ*is*

$$\begin{aligned}\mathfrak{L}(s,u) = {} & [\tfrac{\sin\mathfrak{P}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\cos\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]+ \\ & +\mathfrak{C}_1\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]+\mathfrak{C}_4 e^{\cos\mathfrak{P}s+\mathfrak{C}_3}+ \\ & +\tfrac{u}{\kappa}[-\tfrac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]]]\mathbf{e}_1+ \\ & +[\tfrac{\sin\mathfrak{P}}{\mathfrak{C}_1^2+\cos^2\mathfrak{P}}[-\mathfrak{C}_1\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]+\mathfrak{C}_5 e^{-\cos\mathfrak{P}s-\mathfrak{C}_3}+ \\ & +\tfrac{u}{\kappa}[\tfrac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]-\cos\mathfrak{P}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]]\mathbf{e}_2+ \\ & +[\cos\mathfrak{P}s+\tfrac{u}{\kappa}[\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1 s+\mathfrak{C}_2]-\sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1 s+\mathfrak{C}_2]]+\mathfrak{C}_3]\mathbf{e}_3,\end{aligned} \tag{12}$$

where $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5$ *are constants of integration.*

Proof. Assume that γ be a unit speed non-geodesic general helix. After Theorem 3.2, it is possible to state that

$$\mathbf{T} = \sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]\,\mathbf{e}_1 + \sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]\,\mathbf{e}_2 + \cos\mathfrak{P}\mathbf{e}_3.$$

And, using the first equation of Eq.(8), we have

$$\nabla_{\mathbf{T}}\mathbf{T} = \left(T_1' + T_1T_3\right)\mathbf{e}_1 + \left(T_2' - T_2T_3\right)\mathbf{e}_2 + \left(T_3' - T_1^2 + T_2^2\right)\mathbf{e}_3.$$

This can be rewritten as

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{T} &= [-\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s + \mathfrak{C}_2] + \cos\mathfrak{P}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s + \mathfrak{C}_2]]\mathbf{e}_1 + \\ &\quad +[\frac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s + \mathfrak{C}_2] - \cos\mathfrak{P}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s + \mathfrak{C}_2]]\mathbf{e}_2 + \\ &\quad +[\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1 s + \mathfrak{C}_2] - \sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1 s + \mathfrak{C}_2]]\mathbf{e}_3.\end{aligned}$$

By the use of Frenet formulas and the above equation, it is achieved

$$\begin{aligned}\mathbf{N} &= \tfrac{1}{\kappa}[-\tfrac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s + \mathfrak{C}_2] + \cos\mathfrak{P}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s + \mathfrak{C}_2]]\mathbf{e}_1 + \\ &\quad + \tfrac{1}{\kappa}[\tfrac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s + \mathfrak{C}_2] - \cos\mathfrak{P}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s + \mathfrak{C}_2]]\mathbf{e}_2 + \\ &\quad + \tfrac{1}{\kappa}[\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1 s + \mathfrak{C}_2] - \sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1 s + \mathfrak{C}_2]]\mathbf{e}_3.\end{aligned} \tag{13}$$

□

Combining Eq.(13) and Eq.(11), it is obtained Eq.(12).

Theorem 4.2. *Let $\gamma : I \longrightarrow \mathfrak{Sol}^3$ be a unit speed non-geodesic general helix and $\mathfrak{L}$its normal ruled surface on $\mathfrak{Sol}^3$. Then, the parametric equations of $\mathfrak{L}$ are*

$$\begin{aligned}x_{\mathfrak{L}}(s,u) &= \exp[-\cos\mathfrak{P}s - \tfrac{\mathfrak{u}}{\kappa}[\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1 s + \mathfrak{C}_2] - \sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1 s + \mathfrak{C}_2]] - \mathfrak{C}_3] \\ &\quad [\tfrac{\sin\mathfrak{P}}{\mathfrak{C}_1^2 + \cos^2\mathfrak{P}}[-\cos\mathfrak{P}\cos[\mathfrak{C}_1 s + \mathfrak{C}_2] + \mathfrak{C}_1\sin[\mathfrak{C}_1 s + \mathfrak{C}_2]] + \mathfrak{C}_4 e^{\cos\mathfrak{P}s + \mathfrak{C}_3} + \\ &\quad + \tfrac{\mathfrak{u}}{\kappa}[-\tfrac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s + \mathfrak{C}_2] + \cos\mathfrak{P}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s + \mathfrak{C}_2]]], \\ y_{\mathfrak{L}}(s,u) &= \exp[\cos\mathfrak{P}s + \tfrac{\mathfrak{u}}{\kappa}[\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1 s + \mathfrak{C}_2] - \sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1 s + \mathfrak{C}_2]] + \mathfrak{C}_3] \\ &\quad [\tfrac{\sin\mathfrak{P}}{\mathfrak{C}_1^2 + \cos^2\mathfrak{P}}[-\mathfrak{C}_1\cos[\mathfrak{C}_1 s + \mathfrak{C}_2] + \cos\mathfrak{P}\sin[\mathfrak{C}_1 s + \mathfrak{C}_2]] + \mathfrak{C}_5 e^{-\cos\mathfrak{P}s - \mathfrak{C}_3} + \\ &\quad + \tfrac{\mathfrak{u}}{\kappa}[\tfrac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s + \mathfrak{C}_2] - \cos\mathfrak{P}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s + \mathfrak{C}_2]]], \\ z_{\mathfrak{L}}(s,u) &= [\cos\mathfrak{P}s + \tfrac{\mathfrak{u}}{\kappa}[\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1 s + \mathfrak{C}_2] - \sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1 s + \mathfrak{C}_2]] + \mathfrak{C}_3],\end{aligned} \tag{14}$$

where $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5$ are constants of integration.

Proof. Using the relations Eq.(4) and Eq.(12), it is achieved Eq.(14). This completes the proof. □

Similarly, an example of the obtained parametric equations for Eq.(14) is illustrated in Fig.2:

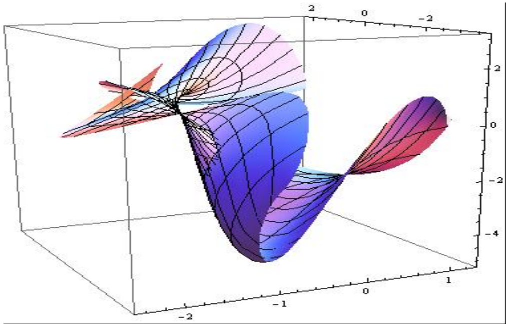

Figure 2.

Thus, the following statement is demonstrated:

Corollary 4.1. *Let $\gamma : I \longrightarrow \mathfrak{Sol}^3$ be a unit speed non-geodesic general helix and $\mathfrak{L}$ its normal ruled surface in Sol space. Then, unit normal of normal ruled surface of γ is*

$$\begin{aligned}
&\mathbf{n}_{\mathfrak{L}}=[\mathcal{M}[\tfrac{1}{\kappa}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2][\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1 s+\mathfrak{C}_2]-\sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1 s+\mathfrak{C}_2]-\\
&-\tfrac{1}{\kappa}\cos\mathfrak{P}[\tfrac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]-\cos\mathfrak{P}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]]-\\
&-\mathcal{N}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]]\mathbf{e}_1+\\
&+[-\mathcal{M}[\tfrac{1}{\kappa}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2][\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1 s+\mathfrak{C}_2]-\sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1 s+\mathfrak{C}_2]-\\
&-\tfrac{1}{\kappa}\cos\mathfrak{P}[-\tfrac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]]]- \qquad (15)\\
&-\mathcal{N}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]\mathbf{e}_2+\\
&+[\mathcal{M}[\tfrac{1}{\kappa}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2][\tfrac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]-\cos\mathfrak{P}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]-\\
&-\tfrac{1}{\kappa}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2][-\tfrac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]]]-\\
&-\mathcal{N}\cos\mathfrak{P}]\mathbf{e}_3,
\end{aligned}$$

where $\mathfrak{C}_1, \mathfrak{C}_2$ are constants of integration and

$$\mathcal{M}=\frac{1-u\kappa}{\sqrt{(1-u\kappa)^2+u^2\tau^2}},\ \mathcal{N}=\frac{u\tau}{\sqrt{(1-u\kappa)^2+u^2\tau^2}}.$$

Proof. Assume that $\mathbf{n}_{\mathfrak{L}}$ is the standard unit normal vector field on normal ruled surface defined by

$$\mathbf{n}_{\mathfrak{L}}=\frac{\mathfrak{L}_s\wedge\mathfrak{L}_u}{|g_{\mathfrak{Sol}^3}(\mathfrak{L}_s\wedge\mathfrak{L}_u,\mathfrak{L}_s\wedge\mathfrak{L}_u)|^{\frac{1}{2}}}. \qquad (16)$$

So it is evident that

$$
\begin{aligned}
\mathbf{B}=&[\tfrac{1}{\kappa}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2][\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1 s+\mathfrak{C}_2]-\sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1 s+\mathfrak{C}_2]-\\
&-\tfrac{1}{\kappa}\cos\mathfrak{P}[\tfrac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]-\cos\mathfrak{P}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]]]\mathbf{e}_1-\\
&-[\tfrac{1}{\kappa}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2][\sin^2\mathfrak{P}\sin^2[\mathfrak{C}_1 s+\mathfrak{C}_2]-\sin^2\mathfrak{P}\cos^2[\mathfrak{C}_1 s+\mathfrak{C}_2]-\\
&-\tfrac{1}{\kappa}\cos\mathfrak{P}[-\tfrac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]]]\mathbf{e}_2+\\
&+[\tfrac{1}{\kappa}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2][\tfrac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]-\cos\mathfrak{P}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]-\\
&-\tfrac{1}{\kappa}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2][-\tfrac{1}{\mathfrak{C}_1}\sin\mathfrak{P}\sin[\mathfrak{C}_1 s+\mathfrak{C}_2]+\cos\mathfrak{P}\sin\mathfrak{P}\cos[\mathfrak{C}_1 s+\mathfrak{C}_2]]]\mathbf{e}_3.
\end{aligned}
\tag{17}
$$

Combining Eq.(16) and Eq.(17), Eq.(15) is obtained. Thus the proof is completed. □

References

[1] Bilici, M., Caliskan, M., (2009), On the involutes of the space-like curve with a time-like binormal in Minkowski 3-space, Int. Math. Forum 4, pp.1497–1509.

[2] Blair, D.E., (1976), Contact Manifolds in Riemannian Geometry, Lecture Notes in Mathematics, Springer-Verlag, 509, Berlin-New York.

[3] Cleave, J.P., (1980), The form of the tangent developable at points of zero torsion on space curves, Math.Proc.Camb.Phil., 88, pp.403–407.

[4] Dimitric, I., (1992), Submanifolds of $\mathbb{E}^m$ with harmonic mean curvature vector, Bull. Inst. Math. Acad. Sinica, 20, pp.53–65.

[5] Ekmekci, N., Ilarslan, K., (2003), Null general helices and submanifolds, Bol. Soc. Mat. Mexicana, 9(2), pp.279-286.

[6] Izumiya, S., Pei, D., Sano, T., (2000), The lightcone Gauss map and the lightcone developable of a spacelike curve in Minkowski 3-space, Glasgow Math. J., 42, pp.75-89.

[7] Körpınar, T., Turhan, E., (2012), On spacelike biharmonic slant helices according to bishop frame in the Lorentzian group of rigid motions $\mathbb{E}(1,1)$, Bol. Soc. Paran. Mat., 30(2), pp.91–100.

[8] Kula, L., Yayli, Y., (2005), On slant helix and its spherical indicatrix, Applied Mathematics and Computation, 169, pp.600-607.

[9] Lancret, M.A., (1806), Memoire sur les courbes ‘a double courbure, Memoires presentes alInstitut, 1, pp.416-454.

[10] Loubeau, E., Montaldo, S., (2004), Biminimal immersions in space forms, preprint, 1, math.DG/0405320.

[11] Ou, Y., Wang, Z., (2008), Linear biharmonic maps into Sol, Nil and Heisenberg Spaces, Mediterr. j. math., 5, pp.379–394.

[12] Struik, D.J., (1988), Lectures on Classical Differential Geometry, Dover, New-York.

[13] Takahashi, T., (1977), Sasakian ϕ-symmetric spaces, Tohoku Math. J., 29, pp.91-113.

[14] Turhan, E., Körpınar, T., (2013), Parametric equations of general helices in the sol space $\mathfrak{Sol}^3$, Bol. Soc. Paran. Mat., 31(1), pp.99–104.

[15] Varady, T., Martin, R.R., Cox, J., (1997), Reverse engineering of geometric models-an introduction, Computer Aided Design., 29, pp.255-268.

Mahmut Ergüt, for the photograph and biography, see TWMS J. Pure Appl. Math., V.1, N.1, 2010, p.85.

Talat Körpinar, Essin Turhan, for the photographs and biographies, see TWMS J. Pure Appl. Math., V.2, N.2, 2011, p.255.