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NUMERICAL SOLUTION OF A BOUNDARY VALUE PROBLEM FOR A SECOND ORDER FUZZY DIFFERENTIAL EQUATION*

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Abstract. Two-point boundary value problem is considered for a second order fuzzy differential equation. This problem is reformulated to four different system of crisp boundary value problems, when fuzzy derivative is considered as a generalization of the H- derivative [1]. So, this problem has four different solutions. Numerical solution of each system is obtained by applying the shooting method. Numerical examples to illustrate the method are presented.

Keywords: boundary value problem, second order fuzzy differential equations, generalized differentiability, Shooting method.

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1. Introduction

In many models of physical processes some quantities, for example, boundary values can be uncertain and they are modeled by fuzzy numbers or fuzzy functions. Then the solution of model can be fuzzy function and if the equation of the model has differentials, then we deal with boundary value for fuzzy differential equations (FDEs).

Different approaches are used for solving fuzzy differential equations. The first approach is based on Zadeh's extension principle [32]. In this approach the associated crisp problem is solved and in the obtained solution the boundary values are substituted instead of the real constant. The second approach the problem is solved by writing in the form of a family of differential inclusion [1, 10, 12, 13, 15, 16, 25, 27].

In the third approach it is assumed that the derivatives in the equation are generalized in H-derivative [29] form or in stronlgy generalized H-derivative form [31].

But its known that the approach used H-derivative has the drawback that it leads to solution which have increasing length of their support [6, 23]. To resolve these diffucilties authors in [31] introduced the concept of generalized differentiability. First order FDEs under strongly generalized derivatives are considered in [5]. In [28] the Euler method was applied for solving initial value problem for FDEs. The authors in [2, 14] develop four-stage order Runge-Kutta methods for FDEs. Numerical methods such as Adams and Nystörm methods and predictor-corrector methods for solving FDEs presented in [3, 20, 21]. Numerical method for a boundary value problem for a linear second order FDEs was considered in [8]. In [18] a boundary value problem for FDEs by using a generalized differentiability was considered and new a new concept

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of solutions was presented. In this paper, we propose a numerical algorithm for finding the solutions for boundary value problem for second order FDEs in the form

$$y''(t) = G(t, y(t), y'(t)), \qquad y(0) = \gamma, \qquad y(l) = \lambda, \tag{1}$$

where $t\epsilon[0,l)$, γ and λ with $[\gamma]^\alpha = [\underline{\gamma}_\alpha, \overline{\gamma}_\alpha]$ and $[\lambda]^\alpha = [\underline{\lambda}_\alpha, \overline{\lambda}_\alpha]$ are fuzzy numbers. The paper is organized as follow. In Section 2, we present the basic defination and useful theoretical information. Boundary value problem for second-order FDEs under generalized differentiability, we study in Section 3. Numerical algorithm for solving considered problem problem is introduced in Section 4 and in Section 5 we present some examples of numerical solutions to illustrate our algorithm.

2. Preliminaries

We give some definitions and introduce the necessary notation which will be used this article. A fuzzy number is mapping $u : R \to [0,1]$ with the following properties:

1) u is normal, that is there exists an $x_0 \epsilon R$ such that $u(x_0) = 1$.

2) u is fuzzy set convex, that is for $x, y \epsilon R$ and $0 < \lambda \leq 1$:

$$u(\lambda x + (1-\lambda)y) \geq min\{u(x), u(y)\}. \tag{2}$$

3) u is upper semi-continuous on R.

4) the closure of $\{x\epsilon R | u(x) > 0\}$ is compact.

Then E is called the space of fuzzy numbers.

For each $\alpha\epsilon(0,1]$ the α-level set $[u]^\alpha$ of a fuzzy set u is the subset of points $x\epsilon R$ with

$$[u]^\alpha = x\epsilon R : u(x) \geq \alpha. \tag{3}$$

It is clear that α-level set of u is an $[\underline{u}^\alpha, \overline{u}^\alpha]$,where $\underline{u}$ and $\overline{u}$ are called lower and upper branch of u respectively.for $u\epsilon E$ we define the length of u as:

$$len(u) = \sup_\alpha (\underline{u}^\alpha - \overline{u}^\alpha) \tag{4}$$

A fuzzy number in parametric form is presented by an ordered pair of functions $(\underline{u}_\alpha, \overline{u}_\alpha)$, $0 \leq \alpha \leq 1$,satisfying the following properties

(1) $\underline{u}_\alpha$ is a bounded nondecreasing left-continuous function of α over $(0,1]$ and right continuous for $\alpha = 0$.

(2) $\overline{u}_\alpha$ is a bounded nonincreasing left-continuous function on $(0,1]$ and right continuous for $\alpha = 0$.

(3) $\underline{u}_\alpha \leq \overline{u}_\alpha$, $0 \leq \alpha \leq 1$.

The metric on E is defined by the equation

$$D(u,v) = \sup_{0\leq\alpha\leq1} d_H([u]^\alpha, [v]^\alpha),$$

where,

$$d_H([u]^\alpha, [v]^\alpha) = max\{|\underline{u}^\alpha - \underline{v}^\alpha|, |\overline{u}^\alpha - \overline{v}^\alpha|\}$$

is a Hausdorff distance of two interval $[u]^\alpha$ and $[v]^\alpha$. Let u and v be two fuzzy sets. If there exists a fuzzy set w such that $u = v + w$,then w is called the H-difference of u and v and denoted by $u \ominus v$.

(H differentiability or Hukuhara differentiability) Let $I = (0,l)$ and $f : I \to \mathcal{F}$ is a fuzzy function. We say that f is differentiable at $t_0 \in I$ if there exists an element $f'(t_0) \in \mathcal{F}$ such that the limits

$$\lim_{h\to0^+} \frac{f(t_0+h) - f(t_0)}{h} = \lim_{h\to0^+} \frac{f(t_0) - f(t_0-h)}{h} \tag{5}$$

exists and are equal to $f'(t_0)$. Here the limits are taken in the metric space $(\mathcal{F}, D)$. Let $f : I \to \mathcal{F}$ a fuzzy mapping.

- f is (1) differentiable at t_0 if there exists an element $f'(t_0)\epsilon\mathcal{F}$ such that the limits

$$\lim_{h\to 0^+} \frac{f(t_0+h)-f(t_0)}{h} = \lim_{h\to 0^+} \frac{f(t_0)-f(t_0-h)}{h} \tag{6}$$

exists and are equal to $F'(t_0)$.

- f is (2) differentiable at t_0 if there exists an element $f'(t_0)\epsilon\mathcal{F}$ such that the limits

$$\lim_{h\to 0^-} \frac{f(t_0+h)-f(t_0)}{h} = \lim_{h\to 0^-} \frac{f(t_0)-f(t_0-h)}{h} \tag{7}$$

exists and are equal to $f'(t_0)$.

It is obviously that Hukuhara differentiable function has increasing length of support. If the function doesn't has this properties then this function is not H-differentiable. To avoid this difficulty the authors in [31] introduced a more general definition of derivative for fuzzy number valued function in the following form:

Let $f : I \to \mathcal{F}$ be fuzzy function, where $[f(t)]^\alpha = [\underline{f}_\alpha, \overline{f}_\alpha]$, for each $\alpha \in [0,1]$. Then, 1) If f is (1) differentiable in the first form, then $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable functions and $[D_1^1 f(t)]^\alpha = [\underline{f}'_\alpha, \overline{f}'_\alpha]$.

2) If f is (2)-differentiable, then $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable and $[D_2^1 f(t)]^\alpha = [\overline{f}'_\alpha, \underline{f}'_\alpha]$.

Now let fuzzy function f is (1) or (2) differentiable, then the first derivative D_1^1 for $D_2^1 f$ might be (n)-differentiable $(n = 1,2)$ and there are four possibilities $D_1^1(D_1^1 f(t))$, $D_2^1(D_1^1 f(t))$, $D_1^1(D_2^1 f(t))$ and $D_2^1(D_2^1 f(t))$. The second derivatives $D_n^1(D_m^1 f(t))$ are denoted by $D_{n,m}^2 f(t)$ for $n, m = 1,2$. Similar to Theorem 2.1, in [19] authors get following results for the second derivatives.

([19]) Let $D_1^1 f : I \to \mathcal{F}$ or $D_2^1 f : I \to \mathcal{F}$ be fuzzy functions, where $[f(t)]^\alpha = [\underline{f}^\alpha(t), \overline{f}^\alpha(t)]$ for $\forall\alpha \in [0,1]$. Then, 1) If $D_1^1 f$ is (1) differentiable, then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and $[D_{1,1}^2 f(t)]^\alpha = [\underline{f}''_\alpha, \overline{f}''_\alpha]$.

2) If $D_1^1 f$ is (2) differentiable, then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and $[D_{1,2}^2 f(t)]^\alpha = [\overline{f}''_\alpha, \underline{f}''_\alpha]$.

3) If $D_2^1 f$ is (2) differentiable, then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and $[D_{2,1}^2 f(t)]^\alpha = [\overline{f}''_\alpha, \underline{f}''_\alpha]$.

4) If $D_2^1 f$ is (2) differentiable, then $\underline{f}'_\alpha$ and $\overline{f}'_\alpha$ are differentiable functions and $[D_{2,2}^2 f(t)]^\alpha = [\underline{f}''_\alpha, \overline{f}''_\alpha]$.

Proof. (see [19]).

3. Fuzzy boundary value problem (FBVP)

There are four different system, which are denoted as (m,n) $(m,n = 1,2)$ to the system (1)

$(1,1)$-system

$$\begin{cases} \underline{y}''_\alpha(t) = F(t, \underline{y}_\alpha(t), \underline{y}'_\alpha(t)) \\ \overline{y}''_\alpha(t) = F(t, \overline{y}_\alpha(t), \overline{y}'_\alpha(t)) \\ \underline{y}_\alpha(0) = \underline{\gamma}_\alpha, \overline{y}_\alpha(0) = \overline{\gamma}_\alpha \\ \underline{y}_\alpha(\ell) = \underline{\lambda}_\alpha, \overline{y}_\alpha(\ell) = \overline{\lambda}_\alpha, \end{cases} \tag{8}$$

(1, 2)-system

$$\begin{cases} \overline{y}''_\alpha(t) = F(t, \underline{y}_\alpha(t), \underline{y}'_\alpha(t)) \\ \underline{y}''_\alpha(t) = F(t, \overline{y}_\alpha(t), \overline{y}'_\alpha(t)) \\ \underline{y}_\alpha(0) = \underline{\gamma}_\alpha, \overline{y}_\alpha(0) = \overline{\gamma}_\alpha \\ \underline{y}_\alpha(\ell) = \underline{\lambda}_\alpha, \overline{y}_\alpha(\ell) = \overline{\lambda}_\alpha, \end{cases} \tag{9}$$

(2, 1)-system

$$\begin{cases} \overline{y}''_\alpha(t) = F(t, \underline{y}_\alpha(t), \overline{y}'_\alpha(t)) \\ \underline{y}''_\alpha(t) = F(t, \overline{y}_\alpha(t), \underline{y}'_\alpha(t)) \\ \underline{y}_\alpha(0) = \underline{\gamma}_\alpha, \overline{y}_\alpha(0) = \overline{\gamma}_\alpha \\ \underline{y}_\alpha(\ell) = \underline{\lambda}_\alpha, \overline{y}_\alpha(\ell) = \overline{\lambda}_\alpha, \end{cases} \tag{10}$$

(2, 2)-system

$$\begin{cases} \overline{y}''_\alpha(t) = F(t, \overline{y}_\alpha(t), \underline{y}'_\alpha(t)) \\ \underline{y}''_\alpha(t) = F(t, \underline{y}_\alpha(t), \overline{y}'_\alpha(t)) \\ \underline{y}_\alpha(0) = \underline{\gamma}_\alpha, \overline{y}_\alpha(0) = \overline{\gamma}_\alpha \\ \underline{y}_\alpha(\ell) = \underline{\lambda}_\alpha, \overline{y}_\alpha(\ell) = \overline{\lambda}_\alpha. \end{cases} \tag{11}$$

We choose the type of solution and translate problem (1) to the corresponding system of boundary value problem. We can construct solution of the fuzzy boundary value problem (1).

4. Algorithm of numerical solution

In general eq. (8) – (11) can be written as follows:

$$\begin{cases} \underline{y}''_\alpha(t) = H(t, \overline{y}_\alpha(t), \underline{y}_\alpha(t), \overline{y}'_\alpha(t), \underline{y}'_\alpha(t)) \\ \overline{y}''_\alpha(t) = H(t, \overline{y}_\alpha(t), \underline{y}_\alpha(t), \overline{y}'_\alpha(t), \underline{y}'_\alpha(t)) \\ \underline{y}_\alpha(0) = \underline{\gamma}_\alpha, \overline{y}_\alpha(0) = \overline{\gamma}_\alpha \\ \underline{y}_\alpha(\ell) = \underline{\lambda}_\alpha, \overline{y}_\alpha(\ell) = \overline{\lambda}_\alpha. \end{cases} \tag{12}$$

We denote $z = \overline{y}$, $x = \underline{y}$ and eq.(12) equation can be reformulated as follows:

$$\begin{cases} x'' = H(t, z, x, z', x') \\ z'' = H(t, z, x, z', x') \\ x(0) = \underline{\gamma}_\alpha, z(0) = \overline{\gamma}_\alpha \\ x(l) = \underline{\lambda}_\alpha, z(l) = \overline{\lambda}_\alpha. \end{cases} \tag{13}$$

For solving the system eq.(13) we apply Gauss iteration method

$$\begin{cases} x''^{s+1} = H(t, z^s, x^s, z'^s, x'^s) \\ z''^{s+1} = H(t, z^s, x^{s+1}, z'^s, x'^{s+1}) \\ z(0)^{s+1} = \underline{\gamma}_\alpha, z^{s+1}(l) = \overline{\lambda}_\alpha \\ x^{s+1}(0) = \underline{\gamma}_\alpha, x^{s+1}(l) = \overline{\lambda}_\alpha. \end{cases} \tag{14}$$

Equations in eq.(14) at iteration step is solved by using Shooting method.

4.1. **Solution Algorithm.** Based on the arguments above, we propose the following algorithm to solve the problem (1) :

1. Enter problem data: Value of accuracy, ε boundary values; Set iteration counter $s = 0$.

2. Form the initial conditions; x^0, z^0;

3. Form the boundary conditions;$x^{s+1}(0), x^{s+1}(l), z^{s+1}(0), z^{s+1}(l)$;

4. Solve the first equation of (14) CALL Shooting Method;

5. Solve the second equation of (14) CALL Shooting Method;

6. Test Converge
If $\| x^{s+1} - x^s \| < \varepsilon$ AND $\| z^{s+1} - z^s \| < \varepsilon$.

THEN STOP

ELSE $s = s + 1$; go to 4.

5. Numerical examples

Example: In this paper we consider fuzzy boundary value problem (FBVP): Solve the FBVP:

$$\begin{cases} y'' = y' + 3y + f(t) \\ y(0,\alpha) = [\frac{2}{9}(\alpha - 1), \frac{2}{9}(1-\alpha)] \\ y(1,\alpha) = [\frac{2}{9}(\alpha - 1), \frac{2}{9}(1-\alpha)] \end{cases} \tag{15}$$

then solution of $(1,1)-$ system are

$$\begin{cases} \overline{y}(t,\alpha) = [\frac{1}{9}(9t^2 - 9t + 2)](1-\alpha), \\ \underline{y}(t,\alpha) = [\frac{1}{9}(9t^2 - 9t + 2)](\alpha - 1). \end{cases} \tag{16}$$

For $(1,1)-$ system the lower and upper solution of $f(t,\alpha)$ are as follows

$$\begin{cases} \overline{f}(t,\alpha) = (1-\alpha)(3-2t) - 1/3(9t^2 - 9t + 2)(1-\alpha) \\ \underline{f}(t,\alpha) = (\alpha - 1)(3-2t) - 1/3(9t^2 - 9t + 2)(\alpha - 1) \end{cases} \tag{17}$$

In Fig.5.1 the results of numerical solution and exact solution are presented. It is seen that there is a uniformly good approximation to exact solution. We see $\overline{y}(t;\alpha)$ and $\underline{y}(t;\alpha)$ represent a valid fuzzy number when $9t^2 - 9t + 2 \geq 0$, that is for $t \leq \frac{1}{3}$ and $t \geq \frac{2}{3}$. For $t \leq \frac{1}{3}$ we have $(2,2)$ solution, for $t \geq \frac{2}{3}$ we have $(1,1)$ solution and this function is shown unvalid part on this interval $[0.343, 0.657]$.

For $(1,2)$ solution, by solving $(1,2)$ system applying the presented numerical algorithm we get the results that illustrated in Fig.5.2. For $t \leq 0.434$, we have $(1,2)$ solution, for $t \geq 0.444$, we have $(2,1)$ solution and $t \geq 0.99$, we have $(2,2)$ solution. Since we solve $(1,2)$ system then the solution for this problem is on $[0.02, 0.434]$.

For $(2,1)$ solution, by solving $(2,1)$ system we get the results presented in Fig.5.3. For $t \geq 0.535$, we have $(2,1)$ solution and $t \leq 0.525$, we have $(1,2)$ solution. However $(2,2)$-solution does not exist. Because of solving $(2,1)$ system then the solution for this problem is on $[0.525, 1]$.

If we consider this problem under generalized differentiability, $(2,2)$-system gives the same solution for $(1,1)$ solution (solution under the Hukuhara differentiability). we get the results that illustrated in Fig.5.4. For $t \geq 0.596$, we have $(1,1)$ solution, for $t \leq 0.273$, we have $(2,2)$ solution and this function is shown unvalid part on this interval $[0.283, 0.586]$.

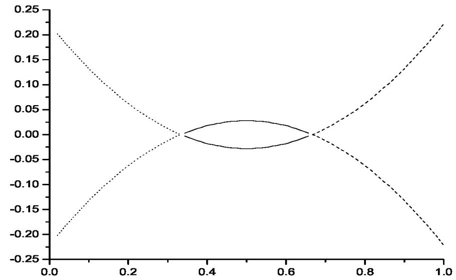

Figure 1. The graph of $(1,1)$-solution and $(2,2)$-solution: $(1,1)$-solution (dash), $(2,2)$-solution (dot), unvalid part (solid).

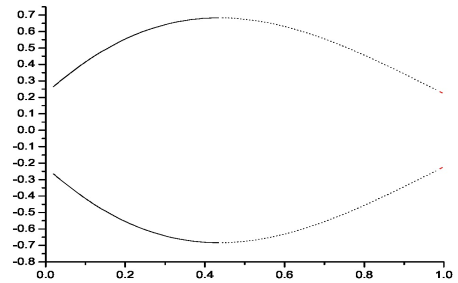

Figure 2. The graph of $(1,2)$-solution and $(2,1)$-solution: $(1,2)$-solution (solid), $(2,1)$-solution (dot), $(2,2)$ solution (dash).

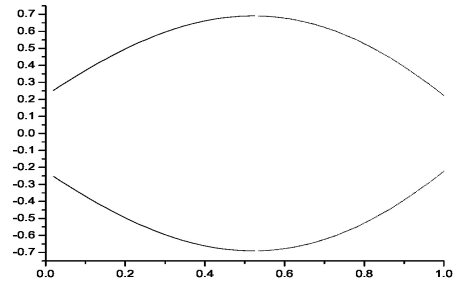

Figure 3. The graph of $(2,1)$-solution and $(1,2)$-solution: $(1,2)$-solution (solid), $(2,1)$-solution (dot).

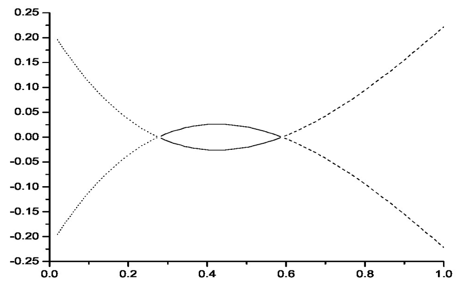

Figure 4. The graph of $(2,2)$-solution and $(1,1)$-solution: $(2,2)$-solution (dot), $(1,1)$-solution (dash),unvalid part (solid).

6. Conclusion

In this paper, there are four different solutions for two-point FBVPs when the fuzzy derivative is regarded as a lateral type of H-derivative. The proposed method was tested on a test example, and has been effective. This method can also be used to solve nonlinear problems with known results on the existence and uniqueness of solutions. This will be the subject of our future work.

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