SOME SUBORDINATION AND SUPERORDINATION RESULTS ASSOCIATED WITH A NEW OPERATOR

M.K. AOUF, A.O. MOSTAFA, A. SHAMANDY, E.A. ADWAN

ABSTRACT. In this paper, we obtain some subordination and superordination results of \( p \)-valent meromorphic functions associated with a new linear operator. Sandwich-type theorem for these \( p \)-valent functions is also obtained.

Keywords: \( p \)-Valent meromorphic functions, subordination, superordination, linear operator.

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1. INTRODUCTION

Let \( H(U) \) be the class of functions analytic in \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) and \( H[a,n] \) be the subclass of \( H(U) \) consisting of functions of the form \( f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots \), with \( H_0 = H[0,1] \) and \( H = H[1,1] \). Let \( \Sigma_p \) denote the class of functions of the form:

\[
f(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p} z^{n-p} \quad (p \in \mathbb{N} = \{1, 2, \ldots \}),
\]

For \( f, F \in H(U) \), the function \( f(z) \) is said to be subordinate to \( F(z) \), or \( F(z) \) is superordinate to \( f(z) \), if there exists a function \( \omega(z) \) analytic in \( U \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 (z \in U) \), such that \( f(z) = F(\omega(z)) \). In such a case we write \( f(z) \prec F(z) \). If \( F \) is univalent, then \( f(z) \prec F(z) \) if and only if \( f(0) = F(0) \) and \( f(U) \subset F(U) \) (see [5] and [6]).

Let \( \phi : \mathbb{C}^2 \times U \to \mathbb{C} \) and \( h(z) \) be univalent in \( U \). If \( p(z) \) is analytic in \( U \) and satisfies the first order differential subordination:

\[
\phi \left( p(z), z p'(z); z \right) \prec h(z),
\]

then \( p(z) \) is a solution of the differential subordination (2). The univalent function \( q(z) \) is called a dominant of the solutions of the differential subordination (2) if \( p(z) \prec q(z) \) for all \( p(z) \) satisfying (2). A univalent dominant \( \bar{q} \) that satisfies \( \bar{q} \prec q \) for all dominants of (2) is called the best dominant. If \( p(z) \) and \( \phi \left( p(z), z p'(z); z \right) \) are univalent in \( U \) and if \( p(z) \) satisfies the first order differential superordination:

\[
h(z) \prec \phi \left( p(z), z p'(z); z \right),
\]

then \( p(z) \) is a solution of the differential superordination (3). An analytic function \( q(z) \) is called a subordinant of the solutions of the differential superordination (3) if \( q(z) \prec p(z) \) for all \( p(z) \) satisfying (3). A univalent subordinant \( \bar{q} \) that satisfies \( q \prec \bar{q} \) for all subordinants of (3) is called

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\(^1\)Department of Mathematics, Faculty of Science, Mansoura University, Egypt

e-mail: mkaouf127@yahoo.com, adelae254@yahoo.com, shamandy16@hotmail.com, eman.a2009@yahoo.com

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where (see [5] and [6]).

For analytic functions \( f(z) \in \sum_p \), given by (1) and \( \phi(z) \in \sum_p \) given by \( \phi(z) = z^{-p} + \sum_{n=1}^{\infty} b_{n-p}z^{n-p} \) \((p \in \mathbb{N})\), the Hadamard product (or convolution) of \( f(z) \) and \( \phi(z) \), is defined by

\[
( f \ast \phi)(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p}b_{n-p}z^{n-p} = (\phi \ast f)(z).
\]

Using (6) and (8), we have

\[
(f \ast \phi)(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p}b_{n-p}z^{n-p} = (\phi \ast f)(z).
\]

Aqlan et al. [1] defined the operator \( Q_{\beta,p}^\alpha : \sum_p \rightarrow \sum_p \) by:

\[
Q_{\beta,p}^\alpha f(z) = \left\{ \begin{align*}
& z^{-p} + \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} \sum_{n=1}^{\infty} \frac{\Gamma(n+\beta)}{\Gamma(n+\beta+\alpha)} a_{n-p}z^{n-p} & (\alpha > 0; \beta > -1; p \in \mathbb{N}; f \in \sum_p) \\
& f(z) & (\alpha = 0; \beta > -1; p \in \mathbb{N}; f \in \sum_p).
\end{align*} \right.
\]

Mostafa [7] used the Aqlan et al. operator and defined the following linear operator \( H_{p,\beta,\mu}^\alpha : \sum_p \rightarrow \sum_p \) as follows:

First put

\[
G_{\beta,p}^\alpha(z) = z^{-p} + \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} \sum_{n=1}^{\infty} \frac{\Gamma(n+\beta)}{\Gamma(n+\beta+\alpha)} z^{n-p} \quad (p \in \mathbb{N})
\]

and let \( G_{\beta,p,\mu}^{\alpha*} \) be defined by

\[
G_{\beta,p}^\alpha(z) = G_{\beta,p}^{\alpha*}(z) = \frac{1}{z^\mu} \quad (\mu > 0; p \in \mathbb{N}).
\]

Then

\[
H_{p,\beta,\mu}^\alpha f(z) = G_{\beta,p}^{\alpha*}(z) \ast f(z) \quad (f \in \sum_p).
\]

Using (6) and (8), we have

\[
H_{p,\beta,\mu}^\alpha f(z) = z^{-p} + \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \sum_{n=1}^{\infty} \frac{\Gamma(n+\beta+\alpha)(\mu)_n}{\Gamma(n+\beta)(\mu)_n} a_{n-p}z^{n-p},
\]

where \((\nu)_n\) denotes the Pochhammer symbol given by

\[
(\nu)_n = \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \begin{cases} 
1 & (n = 0) \\
\nu(\nu+1)...(\nu+n-1) & (n \in \mathbb{N}).
\end{cases}
\]

It is readily verified from (9) that (see [7])

\[
z(H_{p,\beta,\mu}^\alpha f(z))' = (\alpha + \beta)H_{p,\beta,\mu}^{\alpha+1} f(z) - (\alpha + \beta + p)H_{p,\beta,\mu}^\alpha f(z)
\]

and

\[
z(H_{p,\beta,\mu}^\alpha f(z))' = \mu H_{p,\beta,\mu}^{\alpha+1} f(z) - (\mu + p)H_{p,\beta,\mu}^\alpha f(z).
\]

It is noticed that, putting \( \mu = 1 \) in (9), we obtain the operator

\[
H_{p,\beta,1}^\alpha f(z) = H_{p,\beta}^\alpha f(z) = z^{-p} + \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \sum_{n=1}^{\infty} \frac{\Gamma(n+\alpha+\beta)}{\Gamma(n+\beta)} a_{n-p}z^{n-p}.
\]

To prove our results, we need the following definitions and lemmas.
Lemma 1. [5]. Denote by $F$ the set of all functions $q(z)$ that are analytic and injective on $\partial U \setminus E(q)$ where

$$E(q) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty \right\}$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$. Further let the subclass of $F$ for which $q(0) = a$ be denoted by $F(a)$, $F(0) \equiv F_0$ and $F(1) \equiv F_1$.

Definition 1. [6]. A function $L(z, t)$ ($z \in U, t \geq 0$) is said to be a subordination chain if $L(0, t)$ is analytic and univalent in $U$ for all $t \geq 0$, $L(z, 0)$ is continuously differentiable on $[0; 1)$ for all $z \in U$ and $L(z, t_1) < L(z, t_2)$ for all $0 \leq t_1 \leq t_2$.

Lemma 2. [6]. Suppose that the function $H : \mathbb{C}^2 \to \mathbb{C}$ satisfies the condition

$$\Re\left\{ H(iM; t) \right\} \leq 0$$

for all real $s$ and for all $t \leq -n \left(1 + s^2\right)/2$, $n \in \mathbb{N}$. If the function $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \ldots$ is analytic in $U$ and

$$\Re\left\{ H\left(p(z); zq'(z)\right) \right\} > 0 \quad (z \in U),$$

then $\Re\{p(z)\} > 0$ for $z \in U$.

Lemma 3. [4]. Let $\kappa, \gamma \in \mathbb{C}$ with $\kappa \neq 0$ and let $h \in \mathcal{H}(U)$ with $h(0) = c$. If $\Re\{\kappa h(z) + \gamma\} > 0 (z \in U)$, then the solution of the following differential equation:

$$q'(z) + \frac{zq'(z)}{\kappa q(z) + \gamma} = h(z) \quad (z \in U; q(0) = c)$$

is analytic in $U$ and satisfies $\Re\{\kappa q(z) + \gamma\} > 0$ for $z \in U$.

Lemma 4. [5]. Let $p \in F(a)$ and let $q(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$ be analytic in $U$ with $q(z) \neq a$ and $n \geq 1$. If $q$ is not subordinate to $p$, then there exists two points $z_0 = r_0 e^{i\theta} \in U$ and $\zeta_0 \in \partial U \setminus E(q)$ such that

$$q(U_{r_0}) \subset p(U); \quad q(z_0) = p(\zeta_0) \quad \text{and} \quad z_0 p'(z_0) = m\zeta_0 p'\zeta_0 \quad (m \geq n).$$

Lemma 5. [6]. Let $q \in \mathcal{H}[a; 1]$ and $\varphi : \mathbb{C}^2 \to \mathbb{C}$. Also set $\varphi\left(q(z), zq'(z)\right) = h(z)$. If

$$L(z, t) = \varphi\left(q(z), tzq'(z)\right)$$

is a subordination chain and $q \in \mathcal{H}[a; 1] \cap F(a)$, then

$$h(z) < \varphi\left(q(z), zq'(z)\right),$$

implies that $q(z) < p(z)$. Furthermore, if $\varphi\left(q(z), zq'(z)\right) = h(z)$ has a univalent solution $q \in F(a)$, then $q$ is the best subordinator.

In this paper, we investigate several properties of the linear operator $H_{p,\beta,\mu}^\alpha$. 

2. Main Results

Unless otherwise mentioned, we assume throughout this section that $\alpha \geq 0, \beta > -1, \alpha + \beta \neq 0, \mu, \gamma, \eta > 0, p \in \mathbb{N}, z \in U$ and all powers are understood as principle values.

**Theorem 1.** Let $f, g \in \sum_p$ and let

$$
Re \left\{ 1 + \frac{z\phi''(z)}{\phi'(z)} \right\} > -\delta
$$

$$(\phi(z) = (1 - \gamma) \left( z^p H^\alpha_{p,\beta,\mu} f(z) \right)^\eta + \gamma \left( \frac{H^{\alpha+1}_{p,\beta,\mu} f(z)}{H^\alpha_{p,\beta,\mu} f(z)} \right) \left( z^p H^\alpha_{p,\beta,\mu} g(z) \right)^\eta ; z \in U), \quad (14)
$$

where $\delta$ is given by

$$
\delta = 1 + \left( \frac{\eta(\alpha + \beta)}{\gamma} \right)^2 - 1 - \left( \frac{\eta(\alpha + \beta)}{\gamma} \right)^2
$$

$$(15)
$$

Then the subordination condition

$$(1 - \gamma) \left( z^p H^\alpha_{p,\beta,\mu} f(z) \right)^\eta + \gamma \left( \frac{H^{\alpha+1}_{p,\beta,\mu} f(z)}{H^\alpha_{p,\beta,\mu} f(z)} \right) \left( z^p H^\alpha_{p,\beta,\mu} g(z) \right)^\eta <
$$

$$(18)
$$

implies that

$$(z^p H^\alpha_{p,\beta,\mu} f(z))^\eta < (z^p H^\alpha_{p,\beta,\mu} g(z))^\eta
$$

and the function $\left( z^p H^\alpha_{p,\beta,\mu} g(z) \right)^\eta$ is the best dominant.

**Proof.** Define the functions $F(z)$ and $G(z)$ in $U$ by

$$
F(z) = (z^p H^\alpha_{p,\beta,\mu} f(z))^\eta \quad \text{and} \quad G(z) = (z^p H^\alpha_{p,\beta,\mu} g(z))^\eta \quad (z \in U), \quad (18)
$$

and assume, without loss of generality, that $G(z)$ is analytic, univalent on $\bar{U}$ and $G'(\zeta) \neq 0 \quad (|\zeta| = 1).

If not, then we replace $F(z)$ and $G(z)$ by $F(\rho z)$ and $G(\rho z)$, respectively, with $0 < \rho < 1$. These new functions have the desired properties on $\bar{U}$, so we can use them in the proof of our theorem, the results would follow by letting $\rho \to 1$.

We first show that, if

$$
q(z) = 1 + \frac{zG''(z)}{G'(z)} \quad (z \in U), \quad (19)
$$

then

$$
Re \{ q(z) \} > 0 \quad (z \in U).
$$

From (11) and the definition of the functions $G, \phi$, we obtain that

$$
\phi(z) = G(z) + \frac{\gamma}{\eta(\alpha + \beta)} z G'(z).
$$

Differentiating both side of (20) with respect to $z$ yields

$$
\phi'(z) = \left( 1 + \frac{\gamma}{\eta(\alpha + \beta)} \right) G'(z) + \frac{\gamma}{\eta(\alpha + \beta)} z G''(z).
$$

(21)
Combining (19) and (21), we easily get
\[ 1 + \frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{q(z)} + \frac{\eta(\alpha + \beta)}{\gamma} = h(z) \quad (z \in U). \tag{22} \]

It follows from (14) and (22) that
\[ \text{Re} \left\{ h(z) + \frac{\eta(\alpha + \beta)}{\gamma} \right\} > 0 \quad (z \in U). \tag{23} \]

Moreover, by using Lemma 3, we conclude that the differential equation (22) has a solution \( q(z) \in H(U) \) with
\[ h(0) = q(0) = 1. \]
Let
\[ H(u, v) = u + \frac{v}{u + \frac{\eta(\alpha + \beta)}{\gamma}} + \delta, \]
where \( \delta \) is given by (15). From (22) and (23), we obtain
\[ \text{Re} \left\{ H(q(z); zq'(z)) \right\} > 0 \quad (z \in U). \tag{24} \]

To verify the condition
\[ \text{Re} \{ H(iu; v) \} \leq 0 \quad (u \in \mathbb{R}; v \leq \frac{1 + u^2}{2}), \tag{25} \]
we proceed as follows:
\[ \text{Re} \{ H(iu; v) \} = \text{Re} \left\{ iu + \frac{v}{iu + \frac{\eta(\alpha + \beta)}{\gamma}} + \delta \right\} = \frac{\eta(\alpha + \beta)}{\gamma} v + \delta \leq \frac{\sigma(u, \eta, \beta, \alpha, \delta)}{\gamma} u^2 + \left( \eta(\alpha + \beta) \right)^2 \]
where
\[ \sigma(u, \eta, \beta, \alpha, \delta) = \frac{\eta(\alpha + \beta)}{\gamma} - 2\delta u^2 - 2\delta \left( \frac{\eta(\alpha + \beta)}{\gamma} \right)^2 + \frac{\eta(\alpha + \beta)}{\gamma}. \tag{26} \]

For \( \delta \) given by (15), we note that the expression \( \sigma(u, \eta, \beta, \alpha, \delta) \) in (25) is positive, which implies that (24) holds. Thus, by using Lemma 2, we conclude that
\[ \text{Re} \{ q(z) \} > 0 \quad (z \in U). \]
that is, that \( G(z) \) defined by (18) is convex (univalent) in \( U \). Next, we prove that the subordination condition (16) implies that
\[ F(z) \prec G(z), \]
for the functions \( F \) and \( G \) defined by (18). Consider the function \( L(z, t) \) given by
\[ L(z, t) = G(z) + \frac{\gamma (1 + t)}{\eta(\alpha + \beta)} zG'(z) \quad (0 \leq t < \infty; z \in U). \tag{26} \]

We note that
\[ \frac{\partial L(z, t)}{\partial z} \bigg|_{z=0} = G'(0) \left( 1 + \frac{\gamma (1 + t)}{\eta(\alpha + \beta)} \right) \neq 0 \quad (0 \leq t < \infty; z \in U). \]
This show that the function
\[ L(z, t) = a_1(t) z + \ldots, \]
satisfies the condition \( a_1(t) \neq 0 \) for all \( t \geq 0 \) and \( \lim_{t \to \infty} |a_1(t)| = +\infty \). From definition (26) and for all \( t \geq 0 \), we have
Let \( G \) be a function in \( K \). Then, we deduce that

\[
\frac{G(z) + \frac{\gamma(1+t)}{\eta(\alpha+\beta)} zG'(z)}{1 + \frac{\gamma(1+t)}{\eta(\alpha+\beta)}} \leq \frac{|G(z)| + \frac{\gamma(1+t)}{\eta(\alpha+\beta)} |zG'(z)|}{1 + \frac{\gamma(1+t)}{\eta(\alpha+\beta)}}.
\]

(27)

Since the function \( G \) is convex and normalized in \( U \), \( G \in K \), the following well-known growth and distortion sharp inequalities (see [2]) are true:

\[
\frac{r}{1+r} \leq |G(z)| \leq \frac{r}{1-r}, \quad \text{if } |z| \leq r < 1,
\]

\[
\frac{1}{(1+r)^2} \leq |G'(z)| \leq \frac{1}{(1-r)^2}, \quad \text{if } |z| \leq r < 1,
\]

Using the right-hand sides of these inequalities in (27), we deduce that

\[
\frac{|L(z,t)|}{|a_1(t)|} = \frac{r}{1-r} \frac{\gamma(1+t)+(1-r)}{(1-r)^2 \eta(\alpha+\beta) + \gamma(1+t)} \leq \frac{r}{(1-r)^2}, \quad |z| \leq r, t \geq 0,
\]

and thus, the second assumption of Lemma 1 holds. Furthermore,

\[
\Re \left\{ \frac{z\partial L(z,t)}{\partial z} / \partial L(z,t) / \partial t \right\} = \Re \left\{ \frac{\eta(\alpha+\beta)}{\gamma} + (1+t) q(z) \right\} > 0 \quad (0 \leq t < \infty; z \in U).
\]

Therefore, by using Lemma 1, we deduce that \( L(z,t) \) is a subordination chain. It follows from the definition of subordination chain that

\[
\phi(z) = G(z) + \frac{\gamma}{\eta(\alpha+\beta)} zG'(z) = L(z,0),
\]

and

\[
L(z,0) \prec L(z,t) \quad (0 \leq t < \infty),
\]

which implies that

\[
L(\zeta,t) \notin L(U,0) = \phi(U) \quad (0 \leq t < \infty; \zeta \in \partial U).
\]

(28)

If \( F \) is not subordinate to \( G \), by using Lemma 4, we know that there exist two points \( z_0 \in U \) and \( \zeta_0 \in \partial U \) such that

\[
F(z_0) = G(\zeta_0) \quad \text{and} \quad z_0 F'(z_0) = (1+t) \zeta_0 G'(\zeta_0) \quad (0 \leq t < \infty).
\]

(29)

Hence, by virtue of (16), (18), (26) and (29), we have

\[
L(\zeta_0,t) = G(\zeta_0) + \frac{\gamma(1+t)}{\eta(\alpha+\beta)} z_0 G'(\zeta_0) = F(z_0) + \frac{\gamma z_0 F'(z_0)}{\eta(\alpha+\beta)}
\]

\[
= (1-\gamma) \left( z_0^p H_{p,\beta,\mu}^\alpha f(z_0) \right)^\theta + \gamma \left( \frac{H_{p,\beta,\mu}^{\alpha+1} f(z_0)}{H_{p,\beta,\mu}^\alpha f(z_0)} \right) \left( z_0^p H_{p,\beta,\mu}^\alpha f(z_0) \right)^\theta \in \phi(U).
\]

This contradicts (28). Thus, we deduce that \( F \prec G \). Considering \( F = G \), we see that the function \( G \) is the best dominant. This completes the proof of Theorem 1.

Similarly, we can prove the following theorem.

**Theorem 2.** Let \( f, g \in \sum_p \) and let

\[
\Re \left\{ 1 + \frac{z \phi''(z)}{\phi'(z)} \right\} > -\sigma
\]
Next, to arrive at our desired result, we show that
\[
\phi(z) = (1 - \gamma) \left( z^p H^\alpha_{p,\beta,\mu} g(z) \right)^\lambda + \gamma \left( \frac{H^\alpha_{p,\beta,\mu+1} f(z)}{H^\alpha_{p,\beta,\mu} g(z)} \right) \left( z^p H^\alpha_{p,\beta,\mu} g(z) \right)^\lambda; z \in U,
\]
where \( \sigma \) is given by
\[
\sigma = \frac{1 + \left( \frac{\lambda u}{\gamma} \right)^2 - 1 - \left( \frac{\lambda u}{\gamma} \right)^2}{4 \frac{\lambda u}{\gamma}}.
\]
Then the subordination condition
\[
(1 - \gamma) \left( z^p H^\alpha_{p,\beta,\mu} f(z) \right)^\lambda + \gamma \left( \frac{H^\alpha_{p,\beta,\mu+1} f(z)}{H^\alpha_{p,\beta,\mu} f(z)} \right) \left( z^p H^\alpha_{p,\beta,\mu} f(z) \right)^\lambda \\
\preceq (1 - \gamma) \left( z^p H^\alpha_{p,\beta,\mu} g(z) \right)^\lambda + \gamma \left( \frac{H^\alpha_{p,\beta,\mu+1} g(z)}{H^\alpha_{p,\beta,\mu} g(z)} \right) \left( z^p H^\alpha_{p,\beta,\mu} g(z) \right)^\lambda
\]
implies that
\[
\left( z^p H^\alpha_{p,\beta,\mu} f(z) \right)^\lambda \preceq \left( z^p H^\alpha_{p,\beta,\mu} g(z) \right)^\lambda
\]
and the function \( \left( z^p H^\alpha_{p,\beta,\mu} g(z) \right)^\lambda \) is the best dominant.

We now derive the following theorem.

**Theorem 3.** Let \( f, g \in \sum_p \) and let
\[
Re \left\{ 1 + \frac{z \phi''(z)}{\phi'(z)} \right\} > -\delta
\]
\[
(\phi(z) = (1 - \gamma) \left( z^p H^\alpha_{p,\beta,\mu} g(z) \right)^\eta + \gamma \left( \frac{H^\alpha_{p,\beta,\mu+1} g(z)}{H^\alpha_{p,\beta,\mu} g(z)} \right) \left( z^p H^\alpha_{p,\beta,\mu} g(z) \right)^\eta),
\]
where \( \delta \) is given by (15). If the function
\[
(1 - \gamma) \left( z^p H^\alpha_{p,\beta,\mu} f(z) \right)^\eta + \gamma \left( \frac{H^\alpha_{p,\beta,\mu+1} f(z)}{H^\alpha_{p,\beta,\mu} f(z)} \right) \left( z^p H^\alpha_{p,\beta,\mu} f(z) \right)^\eta
\]
is univalent in \( U \) and \( \left( z^p H^\alpha_{p,\beta,\mu} f(z) \right)^\eta \in \mathcal{F} \), then the superordination condition
\[
(1 - \gamma) \left( z^p H^\alpha_{p,\beta,\mu} g(z) \right)^\eta + \gamma \left( \frac{H^\alpha_{p,\beta,\mu+1} g(z)}{H^\alpha_{p,\beta,\mu} g(z)} \right) \left( z^p H^\alpha_{p,\beta,\mu} g(z) \right)^\eta \\
\preceq (1 - \gamma) \left( z^p H^\alpha_{p,\beta,\mu} f(z) \right)^\eta + \gamma \left( \frac{H^\alpha_{p,\beta,\mu+1} f(z)}{H^\alpha_{p,\beta,\mu} f(z)} \right) \left( z^p H^\alpha_{p,\beta,\mu} f(z) \right)^\eta
\]
implies that
\[
\left( z^p H^\alpha_{p,\beta,\mu} g(z) \right)^\eta \preceq \left( z^p H^\alpha_{p,\beta,\mu} f(z) \right)^\eta
\]
and the function \( \left( z^p H^\alpha_{p,\beta,\mu} g(z) \right)^\eta \) is the best subordinant.

**Proof.** Suppose that the functions \( F, G \) and \( q \) are defined by (18) and (19), respectively. By applying the similar method as in the proof of Theorem 1, we get
\[
Re \{ q(z) \} > 0 \quad (z \in U).
\]
Next, to arrive at our desired result, we show that \( G \preceq F \). For this, we suppose that the function \( L(z,t) \) be defined by (26). Since \( G \) is convex, by applying a similar method as in Theorem 1,
we deduce that $L(z, t)$ is subordination chain. Therefore, by using Lemma 5, we conclude that $G \prec F$. Moreover, since the differential equation

$$
\phi(z) = G(z) + \frac{\gamma}{\eta(\alpha + \beta)}zG'(z) = \varphi \left(G(z), zG'(z)\right)
$$

has a univalent solution $G$, it is the best subordinant. This completes the proof of Theorem 3.

Similarly, we can prove the following theorem.

**Theorem 4.** Let $f, g \in \sum_p$ and let

$$
Re \left\{ 1 + \frac{z\phi''(z)}{\phi'(z)} \right\} > -\sigma
$$

$$
\left( \phi(z) = (1 - \gamma) \left(z^p H_{p, \beta, \mu}^{\alpha} g(z)\right)^{\lambda} + \gamma \left(\frac{H_{p, \beta, \mu+1}^{\alpha} f(z)}{H_{p, \beta, \mu}^{\alpha} g(z)}\right) \left(z^p H_{p, \beta, \mu}^{\alpha} g(z)\right)^{\lambda} \right),
$$

where $\sigma$ is given by (30). If the function

$$
(1 - \gamma) \left(z^p H_{p, \beta, \mu}^{\alpha} f(z)\right)^{\lambda} + \gamma \left(\frac{H_{p, \beta, \mu+1}^{\alpha} f(z)}{H_{p, \beta, \mu}^{\alpha} f(z)}\right) \left(z^p H_{p, \beta, \mu}^{\alpha} f(z)\right)^{\lambda}
$$

is univalent in $U$ and $\left(z^p H_{p, \beta, \mu}^{\alpha} f(z)\right)^{\lambda} \in \mathcal{F}$, then the superordination condition

$$
(1 - \gamma) \left(z^p H_{p, \beta, \mu}^{\alpha} g(z)\right)^{\lambda} + \gamma \left(\frac{H_{p, \beta, \mu+1}^{\alpha} g(z)}{H_{p, \beta, \mu}^{\alpha} g(z)}\right) \left(z^p H_{p, \beta, \mu}^{\alpha} g(z)\right)^{\lambda} \prec
$$

$$
(1 - \gamma) \left(z^p H_{p, \beta, \mu}^{\alpha} f(z)\right)^{\lambda} + \gamma \left(\frac{H_{p, \beta, \mu+1}^{\alpha} f(z)}{H_{p, \beta, \mu}^{\alpha} f(z)}\right) \left(z^p H_{p, \beta, \mu}^{\alpha} f(z)\right)^{\lambda}
$$

implies that

$$
\left(z^p H_{p, \beta, \mu}^{\alpha} g(z)\right)^{\lambda} \prec \left(z^p H_{p, \beta, \mu}^{\alpha} f(z)\right)^{\lambda}
$$

and the function $\left(z^p H_{p, \beta, \mu}^{\alpha} g(z)\right)^{\lambda}$ is the best subordinant.

Combining Theorem 1 and Theorem 3, we get the following "sandwich-type result".

**Theorem 5.** Let $f, g_i \in \sum_p (i = 1, 2)$ and let

$$
Re \left\{ 1 + \frac{z\phi_j''(z)}{\phi_j'(z)} \right\} > -\delta
$$

$$
\left( \phi_j(z) = (1 - \gamma) \left(z^p H_{p, \beta, \mu}^{\alpha} g_i(z)\right)^{\eta} + \gamma \left(\frac{H_{p, \beta, \mu+1}^{\alpha} g_i(z)}{H_{p, \beta, \mu}^{\alpha} g_i(z)}\right) \left(z^p H_{p, \beta, \mu}^{\alpha} g_i(z)\right)^{\eta} \right) (j = 1, 2),
$$

where $\delta$ is given by (15). If the function

$$
(1 - \gamma) \left(z^p H_{p, \beta, \mu}^{\alpha} f(z)\right)^{\eta} + \gamma \left(\frac{H_{p, \beta, \mu+1}^{\alpha} f(z)}{H_{p, \beta, \mu}^{\alpha} f(z)}\right) \left(z^p H_{p, \beta, \mu}^{\alpha} f(z)\right)^{\eta}
$$
is univalent in $U$ and $\left(z^pH^\alpha_{p,\beta,\mu} f(z)\right)^\eta \in \mathcal{F}$, then the condition

$$(1 - \gamma) \left(z^pH^\alpha_{p,\beta,\mu} g_1(z)\right)^\eta + \gamma \left(H^{\alpha+1}_{p,\beta,\mu} g_1(z)\right) \left(z^pH^\alpha_{p,\beta,\mu} g_1(z)\right)^\eta <$$

$$< (1 - \gamma) \left(z^pH^\alpha_{p,\beta,\mu} f(z)\right)^\eta + \gamma \left(H^{\alpha+1}_{p,\beta,\mu} f(z)\right) \left(z^pH^\alpha_{p,\beta,\mu} f(z)\right)^\eta <$$

$$< (1 - \gamma) \left(z^pH^\alpha_{p,\beta,\mu} g_2(z)\right)^\eta + \gamma \left(H^{\alpha+1}_{p,\beta,\mu} g_2(z)\right) \left(z^pH^\alpha_{p,\beta,\mu} g_2(z)\right)^\eta$$

implies that

$$\left(z^pH^\alpha_{p,\beta,\mu} g_1(z)\right)^\eta \prec \left(z^pH^\alpha_{p,\beta,\mu} f(z)\right)^\eta \prec \left(z^pH^\alpha_{p,\beta,\mu} g_2(z)\right)^\eta$$

and the functions $\left(z^pH^\alpha_{p,\beta,\mu} g_1(z)\right)^\eta$ and $\left(z^pH^\alpha_{p,\beta,\mu} g_2(z)\right)^\eta$ are, respectively, the best subordinant and the best dominant.

Combining Theorem 2 and Theorem 4, we get the following "sandwich-type result”.

**Theorem 6.** Let $f, g_i \in \sum_p (j = 1, 2)$ and let

$$Re \left\{ 1 + \frac{z\phi''_j(z)}{\phi'_j(z)} \right\} > -\sigma$$

$$\left(\phi_j(z) = (1 - \gamma) \left(z^pH^\alpha_{p,\beta,\mu} g_i(z)\right)^\lambda + \gamma \left(H^{\alpha+1}_{p,\beta,\mu} g_i(z)\right) \left(z^pH^\alpha_{p,\beta,\mu} g_i(z)\right)^\lambda \right),$$

where $\sigma$ is given by (30). If the function

$$(1 - \gamma) \left(z^pH^\alpha_{p,\beta,\mu} f(z)\right)^\lambda + \gamma \left(H^{\alpha+1}_{p,\beta,\mu} f(z)\right) \left(z^pH^\alpha_{p,\beta,\mu} f(z)\right)^\lambda$$

is univalent in $U$ and $\left(z^pH^\alpha_{p,\beta,\mu} f(z)\right)^\lambda \in \mathcal{F}$, then the condition

$$(1 - \gamma) \left(z^pH^\alpha_{p,\beta,\mu} g_1(z)\right)^\lambda + \gamma \left(H^{\alpha+1}_{p,\beta,\mu} g_1(z)\right) \left(z^pH^\alpha_{p,\beta,\mu} g_1(z)\right)^\lambda <$$

$$< (1 - \gamma) \left(z^pH^\alpha_{p,\beta,\mu} f(z)\right)^\lambda + \gamma \left(H^{\alpha+1}_{p,\beta,\mu} f(z)\right) \left(z^pH^\alpha_{p,\beta,\mu} f(z)\right)^\lambda <$$

$$< (1 - \gamma) \left(z^pH^\alpha_{p,\beta,\mu} g_2(z)\right)^\lambda + \gamma \left(H^{\alpha+1}_{p,\beta,\mu} g_2(z)\right) \left(z^pH^\alpha_{p,\beta,\mu} g_2(z)\right)^\lambda$$

implies that

$$\left(z^pH^\alpha_{p,\beta,\mu} g_1(z)\right)^\eta \prec \left(z^pH^\alpha_{p,\beta,\mu} f(z)\right)^\eta \prec \left(z^pH^\alpha_{p,\beta,\mu} g_2(z)\right)^\eta$$

and the functions $\left(z^pH^\alpha_{p,\beta,\mu} g_1(z)\right)^\lambda$ and $\left(z^pH^\alpha_{p,\beta,\mu} g_2(z)\right)^\lambda$ are, respectively, the best subordinant and the best dominant.

**Remark 1.** Putting $\mu = 1$ , in Theorems 1, 3 and 5, we obtain the corresponding results for the operator $H^\alpha_{p,\beta}$ defined in (13).
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References


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Adela Othman Mostafa is an Assistant Professor in Department of Mathematics Faculty of Sciences, Mansoura University, Egypt. She got her Ph.D. (Mathematics) degree in 1990 in University of Mansoura, Egypt. Her research interest is complex analysis.

Ali Shamandy Abd Elwahed is Professor in Department of Mathematics Faculty of Sciences, Mansoura University, Egypt. He got his Ph.D. (Mathematics) degree in 1979 in ELTE, Budapest. His research interest is differential equations.
Eman Ahmed Adwan was graduated in 2007 and got her M. Sc. degree in 2011 and got Ph.D. degree in 2014 in Pure Mathematics from Department of Mathematics, Faculty of Science, Mansoura University, Egypt. Her research interest is complex analysis.