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HOPF HYPERSURFACES IN THE COMPLEX PROJECTIVE SPACE AND THE SASAKIAN SPACE FORM

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Abstract. In this paper, we study isoparametric Hopf hypersurfaces in the complex projective space $\mathbb{C}P^n$ such that the structural vector field is principal and the sectional curvature is weakly constant. Then a similar theory for contact hypersuperfaces of the Sasakian space form is developed.

Keywords: Hopf hypersurfaces, complex projective space, Sasakian manifold.

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1. Introduction

The complex projective space $\mathbb{C}P^n$ can be regarded as the base of the principal fibre bundle associated with a natural action of the group S^1 on the sphere $S^{2n+1} \subset C^{n+1}$. H.B.Lawson [7] (1970) used this idea to study a hypersurface of $\mathbb{C}P^n$ by lifting it to an S^1-invariant hypersurface of the sphere.

An important role plays here the structure vector field of a hypersurface. It is defined by $\xi = JN$, where J is the complex structure and N is the unit normal field. In early investigations, it was found that computations were more tractable when ξ was a principal vector.

A submanifold M of a Riemannian manifold $\widetilde{M}$ is called (extrinsically) homogeneous if there exists a closed subgroup G of the isometry group of $\widetilde{M}$ such that M is an orbit of the action of G on $\widetilde{M}$.

Further, it was observed that ξ is principal for all homogeneous hypersurfaces in $\mathbb{C}P^n$. Later geometric characterizations of this property were found, and the class of Hopf hypersurfaces was defined. The homogeneous hypersurfaces of $\mathbb{C}P^n$ all have constant principle curvatures, and in [6] all hypersurfaces of $\mathbb{C}P^n$ with constant principal curvatures were determined.

The theory of CR submanifolds was developed to include ambient spaces such as locally conformal Kähler manifolds (cf. D.E.Blair and S.Dragomir [3], S.Dragomir and L.Ornea [5], M.H.Shahid [9], quaternionic Kähler manifolds (cf. B.J.Papantoniou and M.H.Shahid [10]). Another version of thought, similar to that concerning Sasakian geometry as an odd-dimensional version of Kählerian geometry (cf. D.E.Blair [2]), considers a submanifold M of an almost contact Riemannian manifold $(\widetilde{M}, (\phi, \xi, \overline{\eta}, \overline{g}))$, carrying an invariant distribution D, $\phi_x(D_x) \subset D_x$ for any $x \in M$, such that the orthogonal complement $D^\perp$ of D in TM is anti-invariant, i.e. $\phi_x D_x^\perp \subseteq T_x^\perp M$ for any $x \in M$. This notion was already used by A.Bejancu and N.Papaghiuc in [1] by using the terminology of semi-invariant submanifolds, any hypersurface M of a Sasakian manifold $\widetilde{M}$ is a contact CR-submanifold.

In this paper we study isoparametric Hopf hypersurfaces of $\mathbb{C}P^n$ with weakly constant holomorphic curvature and prove that these hypersurfaces belong to the list of hypersurfaces given

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in Theorem 2.1 (see Takagi [11]). We also define Hopf hypersurfaces of a Sasakian space form and prove that any such hypersurface with weakly ϕ-section constant curvature has constant principal curvature.

2. Preliminaries

Let $\mathbb{C}^{n+1}$ be the $(n+1)-$dimensional complex space with natural Kähler structure $(J', \langle\ ,\ \rangle)$ and let S^{2n+1} be the unit sphere

$$S^{2n+1} = \{(z^1, \ldots, z^{n+1}) \mid \sum_{i=1}^{n+1} z^i \overline{z}^i = 1\}.$$

Let ψ' be the unit normal vector field to S^{2n+1}. We put $V' = -J'\psi'$, then the integral curve of V' is a great circle $S^1 = \{e^{\sqrt{-1}\theta} \mid 0 \leq \theta < 2\pi\}$. We define a map $S^1 \times S^{2n+1} \to S^{2n+1}$ by

$$(e^{\sqrt{-1}\theta}, \psi) \to e^{\sqrt{-1}\theta}\psi,$$

Then S^1 acts on S^{2n+1} freely and the quotient space of S^{2n+1} is the complex projective space $\mathbb{C}P^n$. Let $p \in S^{2n+1}$ and

$$H_p(S^{2n+1}) = \{X \in T_p(S^{2n+1}) | \langle X, V' \rangle = 0\},$$

Then

$$T_p(S^{2n+1}) = H_p(S^{2n+1}) \oplus \mathrm{span}\{V'_p\},$$

$H_p(S^{2n+1})$ and $\mathrm{span}\{V'_p\}$ are called the *horizontal subspace* and the *vertical subspace* of $T_p(S^{2n+1})$, respectively. By definition, the horizontal subspace $H_p(S^{2n+1})$ is isomorphic to $T_{\pi(p)}(\mathbb{C}P^n)$, where π is the natural projective from S^{2n+1} onto $\mathbb{C}P^n$. Since $H_p(S^{2n+1})$ is $J'-$invariant subspace, so the almost complex structure J can be induced on $T_{\pi(p)}(\mathbb{C}P^n)$.

We define a Riemannian metric g and a connection ∇ in $\mathbb{C}P^n$ respectively by

$$g(X, Y) = g'(X^*, Y^*),$$

$$\nabla_X Y = \pi_*(\nabla'_{X^*} Y^*),$$

where g' is the induced metric S^{2n+1} from $\langle\ ,\ \rangle$ and X^* is a unique horizontal lift of X.

The complex projective space $\mathbb{C}P^n$ with this structure is a Kähler manifold and by Gauss equation we have for the curvature tensor of $\mathbb{C}P^n$

$$\begin{aligned} R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX \\ -g(JX, Z)JY - 2g(JX, Y)JZ. \end{aligned}$$

Suppose that M is a real hypersurface of $\mathbb{C}P^n$ and ψ is the unit normal vector field of M on $\mathbb{C}P^n$. We put $\xi = -J\psi$, then by the Hermitian condition, ξ is a unit tangent vector field on M which is called the *structure vector field* of M. A real hypersurface M is called *a Hopf hypersurface* if ξ is a principal vector field, that is, ξ is an eigenvector of the shape operator A with respect to ψ.

Let M be a submanifold of $\mathbb{C}P^n$ and BM the bundle of unit normal vectors of M. For a sufficiently small real number $t \in \mathbb{R} - \{0\}$, we can define the following immersion,

$$\Phi_t : BM \to \mathbb{C}P^n,$$

$$\psi \to \exp t\psi,$$

where exp denote the exponential mapping of $\mathbb{C}P^n$. This $\Phi_t(BM)$ with induced Riemannian metric from $\mathbb{C}P^n$ ios called the tube of radius t over M in $\mathbb{C}P^n$. Let S^{2n+1} be the unit sphere in $\mathbb{C}P^{n+1} = \mathbb{C}P^{p+1} \oplus \mathbb{C}P^{q+1}$. In S^{2n+1} we choose two sphere, S^{2p+1} and S^{2q+1}, in such a way that they lie respectively in complex subspace $\mathbb{C}P^{p+1}$ and $\mathbb{C}P^{q+1}$ of $\mathbb{C}P^{n+1}$. Then the product $S^{2p+1} \times S^{2q+1}$ is a hypersurface of S^{2n+1} and may be expressed for a fixed t by the following equations

$$\sum_{i=0}^{p} \psi^i \bar{\psi}^i = \cos^2 t, \qquad \sum_{i=p+1}^{n+1} \psi^i \bar{\psi}^i = \sin^2 t.$$

The action of S^1 leaves $S^{2p+1} \times S^{2q+1}$ invariant, and the quotient manifold $S^{2p+1} \times S^{2q+1}/S^1$ is a real hypersurface of $\mathbb{C}P^{n+1}$. We denote this hypersurface by $M^c_{p,q}$. Particularly $M^c_{0,n-1}$ is diffeomorphic with S^{2n-1} and is called geodesic hypersphere.
The manifold $M^c_{n,m}$ is a tube over the totally geodesic complex subspace $\mathbb{C}P^{\frac{n}{2}}$ in $\mathbb{C}P^{\frac{n+p}{2}}$, and the geodesic hypersphere $M^c_{n,0}$ is a tube over the totally geodesic complex hyperplane.

The homogeneous real hypersurfaces in $\mathbb{C}P^{n+1}$ were classified by Ryoichi Takagi [11] in 1973.

Theorem 2.1. *A real hypersurface in $\mathbb{C}P^{n+1}$, $n \geq 2$, is homogeneous if and only if it is congruent to*

1. *A tube around a $k-$dimensional totally geodesic $\mathbb{C}P^k$ in $\mathbb{C}P^{n+1}$ for some $k \in \{0, \cdots, n-1\}$, or*
2. *A tube around the complex quadric $Q^{n-1} = \{[\psi] \in \mathbb{C}P^{n+1} | \psi_0^2 + \ldots + \psi_n^2 = 0\}$ in $\mathbb{C}P^{n+1}$, or*
3. *A tube around the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^k$ into $\mathbb{C}P^{2k+1}$ for some $k \geq 2$, or*
4. *A tube around the Plucker embedding into $\mathbb{C}P^9$ of the complex Grassmann manifold $G_2(\mathbb{C}^5)$ of complex $2-$planes in $\mathbb{C}^5$, or*
5. *A tube around the half spin embedding into $\mathbb{C}P^{15}$ of the Hermitian symmetric space $SO(10)/U(5)$.*

For a homogeneous real hypersurfaces in $\mathbb{C}P^n$ we have $g \in \{2, 3, 5\}$, where g is the number of distinct principal curvatures. Zhen Qi Li [8] proved that $g \in \{2, 3, 5\}$ for all isoparametric real hypersurfaces in $\mathbb{C}P^n$ with constant principal curvature.

Also,Kimura in [6] proved that,

Theorem 2.2. *Let M^n be a isoparametric hypersurface of complex projective space $\mathbb{C}P^n$. Then M^n is homogeneous in $\mathbb{C}P^n$ if and only if it has a constant principal curvature.*

Let $H_p(M)$, $p \in M$ be the $J-$invariant subspace of T_pM. Let $X \in H(M)$ and $H(X) = g(R(X, JX)JX, X)$, then M is said to have a *weakly constant holomorphic curvature* if $H(X)$ is a constant function for any $X \in H(M)$.

A differentiable manifold $\widetilde{M}^{2m+1}$ is said to have an almost contact structure if it admits a (non-vanishing) vector field ξ , a one-form η and a $(1, 1)-$tensor field ϕ satisfying

$$\eta(\xi) = 1 \quad , \quad \phi^2 = -I + \eta \otimes \xi,$$

where I denotes the field of identity transformations of the tangent spaces at all points. These conditions imply that $\phi\xi = 0$ and $\eta \circ \phi = 0$, and that the endomorphism ϕ has rank $2m$ at every point in $\widetilde{M}^{2m+1}$. A manifold $\widetilde{M}^{2m+1}$, equipped with an almost contact structure (ϕ, ξ, η) is called an almost contact manifold and will be denoted by $(\widetilde{M}^{2m+1}, (\phi, \xi, \eta))$.

Suppose that $\widetilde{M}^{2m+1}$ is a manifold carrying an almost contact structure. A Riemannian metric g on $\widetilde{M}^{2m+1}$ satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all vector fields X and Y is called compatible with the almost contact structure, and (ϕ, ξ, η, g) is said to be an almost contact metric structure on $\widetilde{M}^{2m+1}$. It is known that an almost contact manifold always admits at least one compatible metric. Note that putting $Y = \xi$ yields

$$\eta(X) = g(X, \xi)$$

for all vector fields X tangent to $\widetilde{M}^{2m+1}$, which means that η is the metric dual of the characteristic vector field ξ.

A manifold $\widetilde{M}^{2m+1}$ is said to be a contact manifold if it carries a global one-form η such that

$$\eta \wedge (d\eta)^m \neq 0$$

everywhere on M. The one-form η is called the contact form.

A submanifold M of a contact manifold $\widetilde{M}^{2m+1}$ tangent to ξ is called an invariant (resp. anti-invariant) submanifold if $\phi(T_pM) \subset T_pM, \forall p \in M$ (resp. $\phi(T_pM) \subset T_p^{\perp}M, \forall p \in M$).

A submanifold M tangent to ξ of a contact manifold $\widetilde{M}^{2m+1}$ is called a contact CR-submanifold if there exists a pair of orthogonal differentiable distributions D and $D^{\perp}$ on M, such that:

(1) $TM = D \oplus D^{\perp} \oplus \mathbb{R}\xi$, where $\mathbb{R}\xi$ is the $1-$dimensional distribution spanned by ξ;
(2) D is invariant by ϕ, i. e., $\phi(D_p) \subset D_p, \forall p \in M$;
(3) $D^{\perp}$ is anti-invariant by ϕ, i. e., $\phi(D_p^{\perp}) \subset T_p^{\perp}M, \forall p \in M$.

Let $(\widetilde{M}, \phi, \xi, \eta, \widetilde{g})$ be a$(2n+1)$-dimensional contact manifold such that

$$\overline{\nabla}_X \xi = \phi X \qquad , \qquad (\overline{\nabla}_X \phi)Y = \eta(Y)X - \widetilde{g}(X, Y)\xi$$

then $\widetilde{M}$ is called a Sasakian manifold. A Sasakian space form is a Sasakian manifold of constant $\phi-$ sectional curvature and if this is the case, the Riemannian curvature tensor field $\widetilde{R}$ is given by

$$\begin{aligned}\widetilde{R}(X,Y)Z = -\frac{c-1}{4}\{&\eta(Z)[\eta(Y)X - \eta(X)Y] + [\widetilde{g}(Y,Z)\eta(X) - \widetilde{g}(X,Z)\eta(Y)] \\ &+\widetilde{g}(\phi X, Z)\phi Y + 2\widetilde{g}(\phi X, Y)\phi Z - \widetilde{g}(\phi Y, Z)\phi X\} \\ &+\frac{c+3}{4}\{\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y\}\end{aligned}$$

for any $X, Y, Z \in \chi(\widetilde{M})$.

Similarly to Hermitian version, if $g(R(X, \phi X)\phi X, X)$ be constant function for any vector filed X, then M is called weakly constant $\phi-$sectional curvature.

3. Hopf hypersurfaces in $\mathbb{C}P^n$

Let M^{2n+1} be a connected Hopf hypersurface of a complex projective space $\mathbb{C}P^{n+1}$. Let N be a unit normal vector field of M^{2n+1} in $\mathbb{C}P^n$. Then

$$T_pM = H_p(M) \oplus \mathbb{R}\xi$$

for all $p \in M$, where $H_p(M)$ is the horizontal subspace and $\xi = -JN$ is the vertical unit vector field. Since M^{2n+1} is a Hopf hypersurface, the vertical vector field ξ is an eigenvector field of the shape operator A, hence $A\xi = \alpha\xi$.

We begin with result on complex space forms.

Lemma 3.1 (4). *If M^{2n+1} be a connected hypersurface of a complex projective space $\mathbb{C}P^{n+1}$ satisfies the commutative condition $JAX = AJX$ for all tangent vector field X , then ξ is an eigenvector of A with constant eigenvalue and*

$$A^2X - \alpha AX - X + g(\xi, X)\xi = 0.$$

Since A is self adjoint and $H_p(M)$ is invariant subspace under A for any $p \in M$, therefore exist a local frame

$$X_1, \ldots, X_{2n}$$

for $H(M)$ where

$$AX_i = \lambda_i X_i \quad , \quad i = 1, \ldots, 2n.$$

Therefore with set $X = X_i$ in the equation of theorem we have

$$\lambda_i^2 X_i - \alpha\lambda_i X_i - X_i + g(\xi, X)\xi = 0.$$

Because $\{X_i, \xi | i = 1, \ldots, 2n\}$ is linear independent then

$$\lambda_i^2 - \alpha\lambda_i - 1 = 0 \quad , \quad i = 1, , \ldots, 2n.$$

Since α is constant λ_i for all $i = 1, , \ldots, 2n$ is constant. Hence

Theorem 3.1. *Let M^{2n+1} be a connected isoparametric hypersurface of complex projective space $\mathbb{C}P^n$ which satisfies the condition $JAX = AJX$ for all tangent vector fields X. Then M^{2n+1} is one of the hypersurfaces described by Theorem (2.1).*

Corollary 3.1. *Let M^{2n+1} be a connected isoparametric hypersurface of complex projective space $\mathbb{C}P^n$ with satisfies the commutative condition $JAX = AJX$ for all tangent vector field X. Then M^{2n+1} has a weakly constant holomorphic curvature.*

Since A is self adjoint and $H_p(M)$ is an invariant subspace under A for any $p \in M$, there exists a local frame for $H(M)$ which is $A-$invariant. Suppose that this local frame has the form. Suppose the local frame for $H(M)$ be the following form

$$X_1, \ldots . X_n, JX_1, \ldots, JX_n,$$

where

$$AX_i = \lambda_i X_i \quad , \quad AJX_i = \mu_i JX_i \qquad i = 1, \ldots, n.$$

By Gauss equation

$$R(X, Y)Z = \overline{R}(X, Y)Z + g(AY, Z)AX - g(AX, Z)AY,$$

where R and $\overline{R}$ denote the curvature tensors on M^{2n+1} and $\mathbb{C}P^{n+1}$, respectively. Therefore

$$g(R(X_i, JX_i)JX_i, X_i) = 4 + \lambda_i\mu_i.$$

Theorem 3.2. *Let M^{2n+1} be a connected isoparametric Hopf hypersurface of the complex projective space $\mathbb{C}P^n$ with weakly constant holomorphic curvature so that to accept basis as above form. Then M^{2n+1} is one of hypersurfaces listed in Theorem (2.1).*

Proof. First by the assumption we have

$$\lambda_i\mu_i = const. \qquad \forall i = 1, \ldots, n \tag{1}$$

Fix a $i \in \{1, 2, \ldots, n\}$. Now for all tangent vector fields X, Y, Z in Codazzi equation

$$g(\overline{R}(X, Y)Z, N) = g((\nabla_X A)Y - (\nabla_Y A)X, Z) \tag{2}$$

with set $X = X_i$ and $Y = \xi$ we have

$$\begin{aligned} -JX_i &= (\nabla_{X_i} A)\xi - (\nabla_\xi A)X_i \\ &= (X_i\alpha)\xi + \alpha\nabla_{X_i}\xi - A(\nabla_{X_i}\xi) - (\xi\lambda_i)X_i - \lambda_i\nabla_\xi X_i + A(\nabla_\xi X_i). \end{aligned} \tag{3}$$

On the other hand

$$\begin{aligned} \nabla_{X_i}\xi &= -\overline{\nabla}_{X_i}\xi + g(AX_i, \xi) \\ &= -\overline{\nabla}_{X_i}(JN) = -J\overline{\nabla}_{X_i}N \\ &= J(AX_i) = \lambda_i JX_i \end{aligned} \tag{4}$$

so by (3) and (4), we obtain

$$-JX_i = (X_i\alpha)\xi + \alpha\lambda_i JX_i - \lambda_i\mu_i JX_i - (\xi\lambda_i)X_i - \lambda_i\nabla_\xi X_i + A(\nabla_\xi X_i). \tag{5}$$

Suppose

$$\nabla_\xi X_i = \sum_{j=1}^{n} a_j X_j + \sum_{j=1}^{n} b_j JX_j + c\xi. \tag{6}$$

Since $\nabla_\xi\xi = 0$, then in (6)we have $c = 0$. Now by (5)

$$\begin{aligned} (\xi\lambda_i)X_i &+ \sum_{j=1}^{n}\lambda_i a_j X_j + \sum_{j=1}^{n}\lambda_i b_j JX_j - \sum_{j=1}^{n}\lambda_i a_j X_j \\ &- \sum_{j=1}^{n}\mu_j b_j JX_j - \alpha\lambda_i JX_i + \lambda_i\mu_i JX_i - (X_i\alpha)\xi - JX_i = 0 \end{aligned}$$

Since $a_i = 0$ $(g(\nabla_\xi X_i, X_i) = 0)$, then

$$\begin{aligned} (\xi\lambda_i)X_i &+ \sum_{j\neq i}(\lambda_i - \lambda j)a_j X_j + \sum_{j\neq i}(\lambda_i - \mu_j)b_j JX_j \\ &+ (\lambda_i\mu_i - \alpha\lambda_i + \mu_i b_i - \mu_i b - 1)JX_i - (X_i\alpha)\xi = 0. \end{aligned}$$

Since $X_j, JX_j | j = 1, , n$ are linearly independent, we have

$$\xi\lambda_i = 0, \tag{7}$$

$$\lambda_i\mu_i - \alpha\lambda_i + \mu_i b_i - \mu_i b_i - 1 = 0, \tag{8}$$

$$X_i\alpha = 0. \tag{9}$$

Setting $X = JX$ and $Y = U$ in (2) and applying the same method, we get

$$\xi\mu_i = 0 \tag{10}$$

$$\lambda_i\mu_i - \alpha\lambda_i - \lambda_i b_i + \mu_i b_i - 1 = 0, \tag{11}$$

$$JX_i\alpha = 0. \tag{12}$$

Adding (8) to (11), we get

$$2\lambda_i\mu_i - \alpha(\lambda_i + \mu_i) - 2 = 0. \tag{13}$$

Using the covariant derivative of (13) with respect to ξ and the equalities (7) and (10), we obtain

$$(\lambda_i + \mu_i)\xi\alpha = 0.$$

If

$$(\lambda_i + \mu_i)(p) = 0$$

for some $p \in M$ then by (13)

$$\lambda_i^2(p) + 1 = 0$$

and this is impossible. Therefore $\xi\alpha = 0$ and so α is constant. Since $\lambda_i\mu_i$ and α are constant, the relation (13) shows that $\lambda_i + \mu_i$ and hence λ_i and μ_i are constant. This shows that M^{2n+1} is homogeneous and hence by Theorem (2.1) is congruent to one of the following manifolds:

(1) A tube around a $k-$dimensional totally geodesic $\mathbb{C}P^k$ in $\mathbb{C}P^{n+1}$ for some $k \in \{0, \cdots, n-1\}$, or
(2) A tube around the complex quadric $Q^{n-1} = \{[\psi] \in \mathbb{C}Pn+1 | \psi_0^2 + \ldots + \psi_n^2 = 0\}$ in $\mathbb{C}P^{n+1}$, or
(3) A tube around the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^k$ into $\mathbb{C}P^{2k+1}$ for some $k \geq 2$, or
(4) A tube around the Plucker embedding into $\mathbb{C}P^9$ of the complex Grassmann manifold $G_2(\mathbb{C}^5)$ of complex $2-$planes in $\mathbb{C}^5$, or
(5) A tube around the half spin embedding into $\mathbb{C}P^{15}$ of the Hermitian symmetric space $SO(10)/U(5)$.

□

4. Hopf hypersurfaces of a Sasakian space form

Let (M, g) be a real connected hypersurface of $\widetilde{M}(c)$ and N be a unit normal vector field on M. Then we have

$$TM = D \oplus D^{\perp} \oplus \mathbb{R}\xi,$$

where D is a ϕ-invariant subspace and $D^{\perp}$ is the 1-dimensional subspace of TM spanned by $V = \phi(N)$ which is the orthogonal complement of D.

Definition 4.1. *Let A be the shape operator of M and the plan spanned by ξ, V be an invariant subspace of A. Then we call the hypersurface M of $\tilde{M}$ a Hopf hypersurface.*

Lemma 4.1. *Suppose that M is a hypersurface of a Sasakian space form $\tilde{M}(c)$ with the unit normal vector field N on M. Then $\nabla_X V = -\phi AX$ for all $X \in D$.*

Proof. From the Gauss formula and the Sasakian equation we compute

$$\nabla_X V + g(AX, V)N = -\phi AX$$

for all $X \in D$. Considering the tangential and the normal parts, we have $\nabla_X V = -\phi AX$. □

Lemma 4.2. *If M is a hypersurface of a Sasakian space form $\tilde{M}(c)$ with the unit normal vector field N on M, then $A\xi = V$.*

Proof. From the Gauss formula and the Sasakian equation we compute

$$\nabla_V \xi + g(AV, \xi)N = -\phi V = N.$$

Considering the tangential and the normal parts of this relation, we conclude

$$\nabla_V \xi = 0 \quad , \quad g(AV, \xi) = 1, \tag{14}$$

and again we compute

$$\nabla_\xi \xi + g(A\xi, \xi)N = -\phi\xi = 0.$$

Considering the tangential and the normal parts of this relation, we conclude

$$\nabla_\xi \xi = 0 \quad , \quad g(A\xi, \xi) = 0, \tag{15}$$

which implies that $A\xi = V$.

From the Gauss formula and the Sasakian equation with the Weingarten formula and above lemma we compute

$$\nabla_\xi V + g(AV,\xi)N = N,$$

and let $AV = \alpha V + \beta\xi$ we have

$$\nabla_V V + g(AV,V)N = -\phi AV = -\alpha N,$$

considering the tangential and normal part we compute

$$\nabla_\xi V = 0 \quad , \quad \nabla_V V = 0, \tag{16}$$

and $AV = \xi + \alpha V$.

Let M be Hopf hypersurface of $\widetilde{M}(c)$. Since A is self adjoint and D and *span*$\{\xi, V\}$ are invariant under A for any $p \in M$, we may suppose that the local frame for $H(M)$ is of the form

$$X_1, \ldots, X_{n-1}, \phi(X_1), \ldots, \phi(X_{n-1}),$$

for D and $\{W_1, W_2\}$ for *span*$\{\xi, V\}$, where

$$AX_i = \mu_i X_i \quad , \quad A\phi(X_i) = \lambda_i \phi(X_i), \qquad i = 1, \ldots, n-1$$
$$AW_1 = \gamma_1 W_1 \ , \ AW_2 = \gamma_2 W_2.$$

Therefore

$$W_1 = \xi \cos\theta + V \sin\theta,$$
$$W_2 = \xi \sin\theta + V \cos\theta.$$

for some $0 < \theta < \pi/2$. So

$$V = W_1 \sin\theta + W_2 \cos\theta,$$
$$\xi = W_1 \cos\theta - W_2 \cos\theta.$$

□

Lemma 4.3. *Suppose M is hypersurface of Sasakian space form $\widetilde{M}(c)$ then $\gamma_1 = -\tan\theta$ and $\gamma_2 = \cot\theta$.*

Proof. From lemma 4.1 we have

$$AW_1 = A\xi \cos\theta + AV \sin\theta \ = \ -V \cos\theta + AV \sin\theta,$$
$$AW_2 = \ -A\xi \sin\theta + AV \cos\theta = V \sin\theta + AV \cos\theta.$$

Hense

$$V = AW_2 \sin\theta - AW_1 \cos\theta = \gamma_2 W_2 \sin\theta - \gamma_1 W_1 \cos\theta. \tag{17}$$

So we have

$$(\gamma_2 \sin\theta - \cos\theta)W_2 - (\gamma_1 \cos\theta + \sin\theta)W_1 = 0.$$

But since W_1 and W_2 are linearly independent, we have

$$\gamma_1 = -\tan\theta \quad , \quad \gamma_2 = \cot\theta.$$

Hence

$$\gamma_1 = -\tan\theta \quad , \quad \gamma_2 = \cot\theta.$$

So for the eigenvalues γ_1 and γ_2 we have

$$(\gamma_2 - \gamma_1)\cos\theta \sin\theta = 1, \tag{18}$$
$$\gamma_1 \cos^2\theta + \gamma_2 \sin^2\theta = 0. \tag{19}$$

□

Theorem 4.1. *Let M^{2n} be a connected Hopf hypersurface of Sasakian space form $(\tilde{M}^{2n+1}, \phi, \xi, \eta)$ with a weakly constant ϕ-sectional curvature. Then M^{2n} has constant principal curvature*

Proof. By the Gauss equation we have

$$g(R(X_i, \phi X_i)\phi X_i, X_i) = c + \lambda_i \mu_i.$$

Since all ϕ-sectional curvatures of M are constant then

$$\lambda_i \mu_i = const. \qquad \text{for all } 1 \leq i \leq n-1 \tag{20}$$

We set $X = X_i$ $(1 \leq i \leq n-1)$ and $Y = W_j$ $(1 \leq j \leq 2)$ in the Codazzi equation then

$$\begin{aligned} 0 = (\nabla_{X_i} A)W_j - (\nabla_{W_j} A_i)X_i &= (X_i \alpha)W_j + \alpha \nabla_{X_i} W_j - A(\nabla_{X_i} W_j) \\ &\quad -(W_j \lambda_i)X_i - \lambda_i \nabla_{W_j} X_i + A(\nabla_{W_j} X_i). \end{aligned} \tag{21}$$

A direct accounting show that

$$\begin{aligned} \nabla_{X_i} V &= \tan(\overline{\nabla}_{X_i} V) = \tan(\overline{\nabla}_{X_i}(\phi N)) \\ &= \tan((\overline{\nabla}_{X_i} \phi)N + \phi \overline{\nabla}_{X_i} N_) = \tan(\phi(-AX_i)) \\ &= -\mu_i \phi X_i, \end{aligned}$$

and

$$\nabla_{X_i} \xi = \tan(\overline{\nabla}_{X_i} \xi) = \phi X_i,$$

and

$$\begin{aligned} \nabla_{X_i} W_1 = \phi X_i \cos\theta &- \mu_i \phi X_i \sin\theta \\ &+ (X_i(\cos\theta))\xi + (X_i(\sin\theta))V, \end{aligned} \tag{22}$$

and

$$\begin{aligned} \nabla_{X_i} W_2 = -\phi X_i \sin\theta &- \mu_i \phi X_i \cos\theta \\ &- (X_i(\sin\theta))\xi + (X_i(\cos\theta))V. \end{aligned} \tag{23}$$

Also

$$\begin{aligned} \nabla_{W_j} X_i = \nabla_{W_j}(-\phi^2 X_i) &= -\phi^2(\nabla_{W_j} X_i) \\ &= \nabla_{W_j} X_i - g(\nabla_{W_j} X_i, \xi)\xi, \end{aligned}$$

then

$$g(\nabla_{W_j} X_i, \xi) = 0. \tag{24}$$

On the other hand, since

$$\nabla_{W_j} V + g(AW_j, V)N = \overline{\nabla}_{W_j} V = \overline{\nabla}_{W_j}(\phi N) = -\gamma_j \phi W_j,$$

and also $g(AW_j, V)N = -\gamma_j \phi W_j$, then $\nabla_{W_j} V = 0$, so

$$g(\nabla_{W_j} X_i, V) = 0. \tag{25}$$

By (24) and (25) we can suppose

$$\nabla_{W_1} X_i = \sum_{i=1}^{n} a_j X_j + \sum_{j=1}^{n} b_j \phi X_j, \tag{26}$$

$$\nabla_{W_2} X_i = \sum_{i=1}^{n} a'_j X_j + \sum_{j=1}^{n} b'_j \phi X_j. \tag{27}$$

Since the base of $\{X_i, \phi X_i, W_1, W_2 | i = 1, \ldots, n-1\}$ is linear independent, then by (21), (22), (23), (26) and (27) we get

$$W_j \mu_i = 0, \quad j = 1, 2, \tag{28}$$
$$\gamma_1 \cos\theta - \gamma_1 \mu_i \sin\theta - \lambda_i \cos\theta + \mu_i \lambda_i \sin\theta - \mu_i b_i + \lambda_i b_i = 0, \tag{29}$$
$$-\gamma_2 \sin\theta - \gamma_2 \mu_i \cos\theta + \lambda_i \sin\theta + \mu_i \lambda_i \cos\theta - \mu_i b_i' + \lambda_i b_i' = 0, \tag{30}$$
$$X_i \gamma_j = 0, \quad j = 1, 2, \tag{31}$$
$$(\gamma_2 - \gamma_1)((X_i(\cos\theta)\sin\theta - (X_i(\sin\theta))\cos\theta) = 0. \tag{32}$$

We set $X = \phi X_i$ and $Y = W_j$ in the Codazzi equation. Using the similar method, we will have

$$W_j \lambda_i = 0, \quad j = 1, 2 \tag{33}$$
$$\gamma_1 \cos\theta - \gamma_1 \lambda_i \sin\theta - \mu_i \cos\theta + \mu_i \lambda_i \sin\theta + \mu_i b_i - \lambda_i b_i = 0, \tag{34}$$
$$-\gamma_2 \sin\theta - \gamma_2 \lambda_i \cos\theta + \mu_i \sin\theta + \mu_i \lambda_i \cos\theta + \mu_i b_i' - \lambda_i b_i' = 0, \tag{35}$$
$$\phi X_i \gamma_j = 0, \quad j = 1, 2 \tag{36}$$
$$(\gamma_2 - \gamma_1)((\phi X_i(\cos\theta)\sin\theta - (\phi X_i(\sin\theta))\cos\theta) = 0. \tag{37}$$

With set $X = W_1$ and $Y = W_2$ in Codazzi equation, too, we have

$$\begin{aligned} 0 &= (\nabla_{W_1} A) W_2 - (\nabla_{W_2} A_i) W_1 \\ &= (W_1 \gamma_2) W_2 + \gamma_2 \nabla_{W_1} W_2 \\ &\quad - A(\nabla_{W_1} W_2) - (W_2 \gamma_1) W_i - \gamma_1 \nabla_{W_2} X_1 + A(\nabla_{W_2} W_1). \end{aligned} \tag{38}$$

On the other hand, a direct computation shows that

$$\begin{aligned} \nabla_{W_1} W_2 = &- \xi(\cos\theta(\xi(\sin\theta)) + \sin\theta(V(\sin\theta))) \\ &+ V(\cos\theta(\xi(\cos\theta)) + \sin\theta(V(\cos\theta))) \\ &- \cos\theta \sin\theta \nabla_\xi \xi + \cos^2\theta \nabla_\xi V \\ &- \sin^2\theta \nabla_V \xi + \sin\theta \cos\theta \nabla_V V \end{aligned} \tag{39}$$

and

$$\begin{aligned} \nabla_{W_2} W_1 = &- \xi(\sin\theta(\xi(\cos\theta)) - \cos\theta(V(\cos\theta))) \\ &- V(\sin\theta(\xi(\sin\theta)) - \cos\theta(V(\sin\theta))) \\ &- \cos\theta \sin\theta \nabla_\xi \xi + \sin^2\theta \nabla_\xi V \\ &+ \cos^2\theta \nabla_V \xi + \sin\theta \cos\theta \nabla_V V. \end{aligned} \tag{40}$$

From (39) and (40) we have

$$\begin{aligned} \nabla_{W_1} W_2 = &- \xi(\cos\theta(\xi(\sin\theta)) + \sin\theta(V(\sin\theta))) \\ &+ V(\cos\theta(\xi(\cos\theta)) + \sin\theta(V(\cos\theta))), \\ \nabla_{W_2} W_1 = &- \xi(\sin\theta(\xi(\cos\theta)) - \cos\theta(V(\cos\theta))) \\ &- V(\sin\theta(\xi(\sin\theta)) - \cos\theta(V(\sin\theta))). \end{aligned}$$

Then from (38) we have

$$\begin{aligned}[\cos\theta(\xi(\sin\theta)) + \sin\theta(V(\sin\theta))](A\xi - \gamma_2\xi) \\ +[\cos\theta(\xi(\cos\theta)) + \sin\theta(V(\cos\theta))](\gamma_2 V - AV) \\ +[\sin\theta(\xi(\cos\theta)) - \cos\theta(V(\cos\theta))](\gamma_1\xi - A\xi) \\ +[\sin\theta(\xi(\sin\theta)) - \cos\theta(V(\sin\theta))](\gamma_1 V - AV) = 0,\end{aligned}$$

so

$$\begin{aligned}W_1(\gamma_2) - (\gamma_1\sin\theta - \gamma_2\sin\theta)[\sin\theta(\xi(\cos\theta)) - \cos\theta(V(\cos\theta))] \\ +(\gamma_1\cos\theta - \gamma_2\cos\theta)[\sin\theta(\xi(\sin\theta)) - \cos\theta(V(\sin\theta))] = 0 \qquad (41)\end{aligned}$$

and

$$\begin{aligned}W_2(\gamma_1) - (\gamma_2\cos\theta - \gamma_1\cos\theta)[\cos\theta(\xi(\sin\theta)) + \sin\theta(V(\sin\theta))] \\ +(\gamma_2\sin\theta - \gamma_1\sin\theta)[\cos\theta(\xi(\cos\theta)) + \sin\theta(V(\cos\theta))] = 0. \qquad (42)\end{aligned}$$

Now by adding (30) to (35) and (29) to (34) we have

$$2\gamma_2\sin\theta + (\gamma_2\cos\theta - \sin\theta)(\lambda_i + \mu_i) - 2\mu_i\lambda_i\cos\theta = 0, \qquad (43)$$

$$2\gamma_1\cos\theta - (\gamma_1\sin\theta + \cos\theta)(\lambda_i + \mu_i) + 2\mu_i\lambda_i\sin\theta = 0. \qquad (44)$$

By (43) and (44)

$$(\lambda_i + \mu_i)(\gamma_1 + \gamma_2) - 2\lambda_i\mu_i + 2 = 0. \qquad (45)$$

From lemma 4.3 we have

$$W_j(\gamma_1\gamma_2) = 0. \qquad (46)$$

By (45) if $(\lambda_i + \mu_i)(p) = 0$ for some $p \in M$, then $\lambda_i^2(p) = -1$ and this is impossible, we have

$$W_j(\gamma_1 + \gamma_2) = 0. \qquad (47)$$

Therefore

$$W_j(\gamma_1) = W_j(\gamma_2) = 0. \qquad (48)$$

Now by (31) and (36) γ_1 and γ_2 are constant.

From (31), (36) and (18)

$$(\gamma_1 + \gamma_2)X(\lambda_i + \mu_i) = 0.$$

Hence if $\gamma_1 + \gamma_2 = 0$ then by (45) conclude

$$\gamma_1^2 = \gamma_2^2 = 1 \quad , \quad \lambda_i\mu_i = 1.$$

With product of equation (43) to (44) we have

$$(4 - (\lambda_i + \mu_i)^2)(\gamma_1^2 - 2) = 0.$$

Since $\gamma_1^2 - 2 \neq 0$ then

$$(\lambda_i + \mu_i)^2 = 4$$

so $\lambda_i + \mu_i$ are constant.

In other case if $\gamma_1 + \gamma_2 \neq 0$ then

$$X(\lambda_i + \mu_i) = 0,$$

hence, again $\lambda_i + \mu_i$ are constant.

Therefore $\lambda_i, \mu_i, \gamma_1$ and γ_2 are constant for $i = 1, \ldots, n-1$. □

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