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STRUCTURE OF THE GENERAL SOLUTION OF THE MOMENT PROBLEM IN NORMED SPACES AND ITS APPLICATIONS

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Abstract. In this paper, we consider the abstract moment problem in normed spaces and determine the structure of the general solution of this problem. As an example, we consider the control problem for a class of linear distributed-parameter control systems and determine the structure of the general solution of this problem.

Keywords: moment problems, control problems, distributed-parameter systems, normed spaces.

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1. Introduction

Moment problem occupies an important place in the mechanics and mathematics literature from the end of the nineteenth century. Many problems in functional analysis, control theory, probability and statistics can be written as a moment problem [2, 9, 13, 14].
The abstract moment problem in a normed space X is to find a bounded linear functional $f \in X^*$ satisfying

$$\langle \varphi_k, f \rangle = a_k, \quad k = 1, 2, ..., \tag{1}$$

given linearly independent elements $\varphi_1, \varphi_2, ..., \varphi_n, ... \in X$ and the numbers $a_1, a_2, ..., a_n, ...$, which are complex numbers when X is a complex normed space and are real numbers when X is a real normed space. Here, the space X^* is the dual space of the space X and the notation $\langle x, f \rangle$ shows value in $x \in X$ of a linear functional $f \in X^*$. Finding the necessary and sufficient conditions for the existence of such a functional f is called the classical moment problem. The existence of a solution $f \in X^*$ to the problem in (1) has been extensively analyzed. Several necessary and sufficient conditions for the existence can be found in the literature, e.g. [2, 9, 13, 14]. The moment problems and control problems have been studied extensively in the literature [1, 2, 3, 4, 5, 6, 8, 9, 12, 13, 14, 16, 17, 19, 20, 22].
In this paper, we determine the structure of the general solution of the moment problem (1). The method we follow is similar to finding the general solution of linear algebraic and linear differential equations. In addition, we consider homogenous moment problem

$$\langle \varphi_k, f \rangle = 0, \quad k = 1, 2, ... \,. \tag{2}$$

The general solution of the non-homogenous moment problem (1) is written as the sum of a particular solution of the moment problem (1) and a general solution of the homogenous moment problem (2). We also give a construction method of a particular solution of the moment problem (1). Furthermore, we show how to translate a control problem of a class of distributed-parameter

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control systems into a moment problem; and determine the structure of the general solution of this control problem.

2. Structure of the general solution

In this section, we determine the structure of the general solution of the moment problem (1). It is clear that $f \equiv 0$ is the trivial solution of the homogenous moment problem (2). Each non-trivial solution of the homogenous moment problem (2) is called null-solution and any solution of the non-homogenous moment problem (1) is called a particular solution, denoted by f° and f^*, respectively.

Theorem 2.1. *Every solution f of the non-homogenous moment problem (1) can be written as the sum of a special solution f^* and the null-solution f°, that is*

$$f = f^* + f^\circ.$$

Proof. Assume that the functional f is any solution of the moment problem (1) and f^* is a particular solution of the moment problem (1). Then,

$$\langle \varphi_k, f \rangle = a_k, \quad k = 1, 2, ..., \tag{3}$$

$$\langle \varphi_k, f^* \rangle = a_k, \quad k = 1, 2, ... \;. \tag{4}$$

If we subtract the equalities (3) and (4) at a same index k, we find the equality

$$\langle \varphi_k, f - f^* \rangle = 0, \quad k = 1, 2, ... \;. \tag{5}$$

From the equalities (5), it is seen that the difference $f - f^*$ is the solution of the homogenous moment problem (2) corresponding to moment problem (1):

$$f - f^* = f^\circ. \tag{6}$$

From the equality (6), we obtain

$$f = f^* + f^\circ.$$

Thus, the proof is completed.

Proposition 2.1. *There exists a non-zero functional f° satisfying*

$$\langle \varphi_k, f^\circ \rangle = 0, \quad k = 1, 2, ...,$$

if and only if the system $\{\varphi_n : n = 1, 2, \dots\}$ is not a complete system.

Proof. We prove the necessary condition when X is countable based Banach space which is given in [24]. Suppose $\{\varphi_k\}$ is a complete system in X, and the functional f° is a solution of the moment problem (2). Then, an arbitrary element $x \in X$ has a unique expansion

$$x = \sum_{n=1}^{\infty} \alpha_n \varphi_n. \tag{7}$$

If we substitute (7) into the homogenous moment problem (2), we obtain

$$\langle x, f^\circ \rangle = \left\langle \sum_{k=1}^{\infty} \alpha_k \varphi_k, f^\circ \right\rangle = \sum_{k=1}^{\infty} \alpha_k \langle \varphi_k, f^\circ \rangle = 0.$$

Since this is valid for each $x \in X$, we have $f^\circ \equiv 0$. This contradiction proves necessary condition. Sufficient condition follows directly from Hahn-Banach Theorem.
The following theorem can be easily proved, hence the proof is omitted.

Theorem 2.2. *Assume that the functionals $f_1^\circ, f_2^\circ, ..., f_n^\circ \in X^*$ are linearly independent solutions of the homogenous moment problem (2). Then also the linear functional f° defined by equality*

$$f^\circ = c_1 f_1^\circ + c_2 f_2^\circ + \cdots + c_n f_n^\circ,$$

where $c_1, c_2, ..., c_n$ are arbitrary constants, is a solution of the homogenous moment problem (2).

By Theorem 2.2., the general solution f of the non-homogenous moment problem (1) can be defined as

$$f = f^* + \sum_i c_i f_i^\circ,$$

where c_i are arbitrary constants.

Now, we assume that the system $f_1, f_2, ..., f_n, ... \in X^*$ is biorthogonal to the system $\varphi_1, \varphi_2, ..., \varphi_n, ...$, that is these systems satisfy the conditions

$$\langle \varphi_k, f_\ell \rangle = \delta_{k\ell} = \begin{cases} 1, & k = \ell, \\ 0, & k \neq \ell. \end{cases} \tag{8}$$

Here, the symbol $\delta_{k\ell}$ is Kronecker constant.

Theorem 2.3. *The linear functional defined by*

$$f^* = a_1 f_1 + a_2 f_2 + \cdots + a_n f_n + \cdots, \tag{9}$$

is a solution of the moment problem (1).

The theorem is easily proved by replacing the equality (9) into the non-homogenous moment problem (1) and using the equalities (8). Thus, the general solution f of the non-homogenous moment problem (1) is denoted as

$$f = \sum_k a_k f_k + \sum_i c_i f_i^\circ, \tag{10}$$

where c_i are arbitrary constants.

Note 2.1. The constants $c_1, c_2, ..., c_n, ...$ in (10) can be used in the problem of minimalization of the functional $J(f)$ which characterizes the control process. The f of the functional $J(f)$ is defined by equation (10). In this case, $J(f)$ becomes a function of the variables $c_1, c_2, ..., c_n, ...$. In the special case, the functional $J(f)$ can be taken as

$$J(f) = ||f||.$$

3. On the control problem for a class of linear distributed-parameter systems

3.1. **Statement of the problem.** Let us consider the following control system given with the equation

$$m\left(\frac{\partial}{\partial t}\right) Q = \ell\left(x, \frac{\partial}{\partial x}\right) Q + b(x) f(t), \tag{11}$$

and the boundary conditions

$$\begin{aligned} R_1(Q) &\equiv \alpha_1 Q'_x(0,t) + \beta_1 Q(0,t) = u_1(t), \quad \alpha_1^2+\beta_1^2 \neq 0, \\ R_2(Q) &\equiv \alpha_2 Q'_x(\ell,t) + \beta_2 Q(\ell,t) = u_2(t), \quad \alpha_2^2+\beta_2^2 \neq 0. \end{aligned} \tag{12}$$

Here,

$$\ell\left(x, \frac{\partial}{\partial x}\right) Q = \frac{\partial}{\partial x}\left(p(x)\frac{\partial Q}{\partial x}\right) + q(x)Q,$$
$$m\left(\frac{\partial}{\partial t}\right) Q = a_0 \frac{\partial^2 Q}{\partial t^2} + a_1 \frac{\partial Q}{\partial t},$$

where the coefficients a_0 and a_1 are real constants. The equation (11) is the vibration equation of a wire with length ℓ when $a_0 = 1$, $a_1 = 0$, the thermal conductivity equation of a wire with length ℓ when $a_0 = 0$, $a_1 = 1$ and the Poisson equation when $a_0 = -1$, $a_1 = 0$. Hereafter, we will focus on these cases. Here, the functions $p(x), q(x), b(x)$ are all defined on $[0, \ell]$. We assume that on $[0, \ell]$, $p(x)$ is positive and two times continuously differentiable, $q(x)$ is continuous, and $b(x)$ is two times continuously differentiable and it satisfies boundary conditions $b(0) = 0$, $b(\ell) = 0$ [7]. We consider $f(t)$, $u_1(t)$ and $u_2(t)$ as control functions.
The control problem for the system (11)-(12) is expressed as follows:
Given a set of initial and boundary conditions of time variable t on a bounded interval $[0, T]$, we want to find functions $f(t)$, $u_1(t)$, $u_2(t)$ in such a way that the system given with the conditions (11)-(12) will be carried to a desired final state. This control parameter triplet $f(t)$, $u_1(t)$, $u_2(t)$ is also the unique solution of the initial-boundary value problem given above. We note that the initial and final conditions are given according to the form of the differential expression $m(\partial/\partial t)$. For example, in the case that $a_0 = 0$, $a_1 \neq 0$ the initial and the final state conditions are given as

$$\begin{aligned} Q(x,0) &= Q_0(x), \quad 0 < x < \ell, \\ Q(x,T) &= Q_1(x), \quad 0 < x < \ell. \end{aligned} \tag{13}$$

However, in the case that $a_0 \neq 0$ the initial and the final state conditions are given as

$$\begin{aligned} Q(x,0) &= Q_0(x), \quad 0 < x < \ell, \\ Q'_t(x,0) &= Q_0^{(1)}(x), \quad 0 < x < \ell, \end{aligned} \tag{14}$$

and

$$\begin{aligned} Q(x,T) &= Q_1(x), \quad 0 < x < \ell, \\ Q'_t(x,T) &= Q_1^{(1)}(x), \quad 0 < x < \ell, \end{aligned} \tag{15}$$

where $Q_0(x)$, $Q_1(x)$, $Q_0^{(1)}(x)$, $Q_1^{(1)}(x)$ are given functions defined on the interval $[0, \ell]$.

Definition 3.1. *The control problem expressed with the equations (11)-(12) is called homogenous control problem if the initial and the final state conditions are homogenous i.e. for* $a_0 = 0$, $a_1 \neq 0$

$$\begin{aligned} Q(x,0) &= Q_0(x) \equiv 0, \quad 0 < x < \ell, \\ Q(x,T) &= Q_1(x) \equiv 0, \quad 0 < x < \ell, \end{aligned}$$

for $a_0 \neq 0$

$$\begin{aligned} Q(x,0) &= Q_0(x) \equiv 0, \quad 0 < x < \ell, \\ Q'_t(x,0) &= Q_0^{(1)}(x) \equiv 0, \quad 0 < x < \ell, \end{aligned}$$

and

$$\begin{aligned} Q(x,T) &= Q_1(x) \equiv 0, \quad 0 < x < \ell, \\ Q'_t(x,T) &= Q_1^{(1)}(x) \equiv 0, \quad 0 < x < \ell. \end{aligned}$$

Otherwise, it is a non-homogenous control problem.

Definition 3.2. *Nontrivial control factor carrying the control system from the zero initial state to the zero final state in a finite time interval is called the zero control factor or zero control.*

Control factor carrying the control system from a given initial state to a given final state in a finite time interval is called finite control [9, 10, 11].

3.2. **Boundary value problems with parameter corresponding to the control systems.** In the control problem or control system (11), (12) and (13) [or (11), (12), (14) and (15)], we apply 1D Fourier transform in the second variable and get

$$\widetilde{Q}(x,\omega) = \int_0^T Q(x,t)e^{-i\omega t}dt.$$

Thus, we obtain the following parametrized boundary value problem

$$\ell\left(x, \frac{\partial}{\partial x}\right)\widetilde{Q} + \lambda \widetilde{Q}(x,\omega) = F(x,\omega), \tag{16}$$

$$\begin{aligned} R_1(\widetilde{Q}) &= \tilde{u}_1(\omega), \\ R_2(\widetilde{Q}) &= \tilde{u}_2(\omega). \end{aligned} \tag{17}$$

Here,

$$\lambda = a_0\omega^2 - a_1 i\omega,$$

$$\begin{aligned} F(x,\omega) =& a_0\left[Q_1^{(1)}(x)e^{-i\omega T} - Q^{(1)}(x) + i\omega\left(Q_1(x)e^{-i\omega T} - Q_0(x)\right)\right] \\ &+ a_1\left[Q_1(x)e^{-i\omega T} - Q_0(x)\right] + b(x)\tilde{f}(x). \end{aligned} \tag{18}$$

We suppose that the functions $\psi_1(x,\omega)$ and $\psi_2(x,\omega)$ are the solutions of boundary value problems

$$\ell\left(x, \frac{\partial}{\partial x}\right)\psi_1 + \lambda\psi_1 = 0, \quad R_1(\psi_1) = 0,$$

$$\ell\left(x, \frac{\partial}{\partial x}\right)\psi_2 + \lambda\psi_2 = 0, \quad R_2(\psi_2) = 0.$$

Then, the solution of equation (16) satisfying boundary conditions (17) is found by

$$\widetilde{Q}(x,\omega) = \frac{\psi_2(x,\omega)}{R_1(\psi_2)}\tilde{u}_1(\omega) + \frac{\psi_1(x,\omega)}{R_2(\psi_1)}\tilde{u}_2(\omega) + \int_0^\ell G(x,\xi,\omega)F(\xi,\omega)d\xi. \tag{19}$$

$G(x,\xi,\omega)$ is the Green function and is expressed in [7] as the following

$$G(x,\xi,\omega) = -\frac{1}{p(0)W(0,\omega)}\begin{cases} \psi_1(x,\omega)\psi_2(\xi,\omega), & 0 \le x \le \xi, \\ \psi_1(\xi,\omega)\psi_2(x,\omega), & \xi \le x \le \ell. \end{cases}$$

Here, the function $W(x,\omega)$ is Wronskii determinant of the functions $\psi_1(x,\omega)$ and $\psi_2(x,\omega)$. Now, let $y_1(x,\omega)$ and $y_2(x,\omega)$ be the solutions of the equation

$$\ell\left(x, \frac{\partial}{\partial x}\right)y + \lambda y = 0,$$

satisfying the conditions

$$y_k^{(s-1)}(0,\omega) = \delta_{k,s} = \begin{cases} 0, & k \neq s, \\ 1, & k = s. \end{cases}$$

Then, we can get the functions $\psi_1(x,\omega)$ and $\psi_2(x,\omega)$ as

$$\psi_1(x,\omega) = \begin{vmatrix} y_1(x,\omega) & y_2(x,\omega) \\ R_1(y_1) & R_1(y_2) \end{vmatrix}, \tag{20}$$

$$\psi_2(x,\omega) = \begin{vmatrix} y_1(x,\omega) & y_2(x,\omega) \\ R_2(y_1) & R_2(y_2) \end{vmatrix}. \tag{21}$$

The equalities

$$R_1(\psi_2) = \Delta(\omega), \quad R_2(\psi_1) = -\Delta(\omega), \quad W(0,\omega) = \Delta(\omega),$$

can easily be shown. Here,

$$\Delta(\omega) = \begin{vmatrix} R_1(y_1) & R_2(y_2) \\ R_2(y_1) & R_2(y_2) \end{vmatrix}.$$

Using these relations, we can write the formula (19) as

$$\widetilde{Q}(x,\omega) = \frac{\psi_2(x,\omega)\tilde{u}_1(\omega) - \psi_1(x,\omega)\tilde{u}_2(\omega) - h(x,\omega)/p(0)}{\Delta(\omega)}.$$

Here,

$$h(x,\omega) = \psi_2(x,\omega)\int_0^x F(\xi,\omega)\psi_1(x,\omega)d\xi + \psi_1(x,\omega)\int_x^\ell F(\xi,\omega)\psi_2(\xi,\omega)d\xi. \tag{22}$$

In our setting, the function $Q(x,t)$ must be compactly supported in the interval $[0,T]$ in the second variable. Moreover, $Q(x,t) \in L^2[0,T]$ for all $x \in [0,\ell]$. Hence, according to Wiener-Paley Theorem, for each x, the function $\widetilde{Q}(x,z)$ is an entire function of exponential type with degree not exceeding T. According to Wiener-Paley Theorem, the converse is also true, i.e., $H(z)$ is an entire function of exponential type, its inverse Fourier transform of is supported by the interval $[0,T]$. Thus, to make the function $Q(x,t)$ finite with respect to time, the functions $\tilde{f}(w)$, $\tilde{u}_1(w)$ and $\tilde{u}_2(w)$ must be selected such that the function $\widetilde{Q}(x,w)$ has the needed analyticity. Recall that the spectral problem

$$\ell\left(x,\frac{\partial}{\partial x}\right)\psi + \lambda\psi = 0, \quad R_1(\psi) = 0, \quad R_2(\psi) = 0,$$

has real characteristic values $\{-\lambda_k\}$ and characteristic functions $\varphi_k(x)$ corresponding to the characteristic values. This characteristic values are enumerated in ascending order [15, 21, 23]:

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_{n+1} < \cdots,$$

and

$$\lim_{n\to\infty} \lambda_n = +\infty.$$

The points that the function $\widetilde{Q}(x,\omega)$ is not regular are exactly the points ω_k, which can be founded from

$$a_0\omega_k^2 - i\omega_k a_1 = -\lambda_k, \quad k = 1,2,\ldots.$$

Without loss of generality, here we assume that characteristic numbers λ_k are not repeated. For certainty, we consider the cases

$$a_0 = 0, \quad a_1 = 1,$$

$$a_0 = 1, \quad a_1 = 0,$$

$$a_0 = -1, \quad a_1 = 0.$$

Corresponding to these cases

$$\omega_k = i\lambda_k,$$

$$\omega_k = i\sqrt{\lambda_k},$$

$$\omega_k = \pm\sqrt{\lambda_k}\ .$$

For the function $\widetilde{Q}(x, \omega)$ be regular at the points ω_k, the equalities

$$\psi_2(x, \omega_k)\tilde{u}_1(\omega_k) - \psi_1(x, \omega_k)\tilde{u}_2(\omega_k) - \frac{1}{p(0)}h(x, \omega_k) = 0, \tag{23}$$

must be satisfied for ω_k, $k = 1, 2, ...$. From (20) and (21), the inequalities

$$\psi_1(x, \omega_k) = -\frac{p(0)}{b_k^0}\varphi_k(x), \quad k = 1, 2, ..., \tag{24}$$

$$\psi_2(x, \omega_k) = \frac{p(0)}{b_k^1}\varphi_k(x), \quad k = 1, 2, ..., \tag{25}$$

are obtained. Here,

$$b_k^0 = \begin{cases} -\dfrac{p(0)}{\alpha_1}\varphi_k(0), & \alpha_1 \neq 0, \quad k = 1, 2, ..., \\ \dfrac{p(0)}{\beta_1}\varphi_k'(0), & \alpha_1 = 0, \quad k = 1, 2, ..., \end{cases}$$

$$b_k^1 = \begin{cases} \dfrac{p(\ell)}{\alpha_2}\varphi_k(\ell), & \alpha_2 \neq 0, \quad k = 1, 2, ..., \\ -\dfrac{p(\ell)}{\beta_2}\varphi_k'(\ell), & \alpha_2 = 0, \quad k = 1, 2, ... \end{cases}$$

By substituting equalities (24) and (25) into (22) we find the equalities

$$h(x, \omega_k) = \frac{p(0)}{b_k^0 b_k^1}\int_0^\ell F(\xi, \omega_k)\varphi_k(\xi)d\xi, \quad k = 1, 2, ... \tag{26}$$

By substituting the (26) into (23) we obtain the equality

$$b_k^0\tilde{u}_1(\omega_k) + b_k^1\tilde{u}_2(\omega_k) - \int_0^\ell F(\xi, \omega_k)\varphi_k(\xi)d\xi = 0, \quad k = 1, 2, ... \tag{27}$$

Using the expression of $F(x, \omega_k)$ defined by (18) in (27), we obtain

$$b_k^0\tilde{u}_1(\omega_k) + b_k^1\tilde{u}_2(\omega_k) + b_k\tilde{f}(\omega_k) = \delta_k, \quad k = 1, 2, ..., \tag{28}$$

where

$$\delta_k = \begin{cases} Q_{1k}e^{-i\omega_k T} - Q_{0k}, & a_0 = 0, \quad a_1 \neq 0, \quad k = 1, 2, ..., \\ Q_{1k}^{(1)}e^{-i\omega_k T} - Q_k^{(1)} + i\omega_k\left(Q_{1k}e^{-i\omega_k T} - Q_{0k}\right), & a_0 \neq 0, \quad k = 1, 2, ..., \\ -\left[Q_{1k}^{(1)}e^{-i\omega_k T} - Q_k^{(1)} + i\omega_k\left(Q_{1k}e^{-i\omega_k T} - Q_{0k}\right)\right], & a_0 \neq 0, \quad k = 1, 2, ..., \end{cases}$$

$$Q_{0k} = \int_0^\ell Q_0(x)\varphi_k(x)dx, ..., Q_{1k}^{(1)} = \int_0^\ell Q_1^{(1)}(x)\varphi_k(x)dx.$$

The condition (28) is the interpolation problem to find the functions $\tilde{f}(\omega)$, $\tilde{u}_1(\omega)$ and $\tilde{u}_2(\omega)$. The equality (28) is a linear algebraic equation with three unknowns, for each number k. We can show the general solution of (28) as

$$\begin{aligned} &\tilde{f}(\omega_k) = c_{k1}, \quad k = 1, 2, ..., \\ &\tilde{u}_1(\omega_k) = c_{k2}, \quad k = 1, 2, ..., \\ &\tilde{u}_2(\omega_k) = -\frac{1}{b_k^1}\left(b_k c_{k1} + b_k^0 c_{k2}\right), \quad k = 1, 2, ..., \end{aligned} \tag{29}$$

where c_{k1} and c_{k2} are arbitrary constants. We find the general solution of homogenous control problem for system (11)-(12) by solving (29), which is the interpolation problem given in [18]. Here, since

$$\begin{aligned} \tilde{u}_1(\omega_k) &= \int_0^T e^{-i\omega_k t} u_1(t)dt, \\ \tilde{u}_2(\omega_k) &= \int_0^T e^{-i\omega_k t} u_2(t)dt, \\ \tilde{f}(\omega_k) &= \int_0^T e^{-i\omega_k t} f(t)dt, \end{aligned}$$

we have carried the control problem considered from equalities (29), to the equivalent moment problem [10]:

$$\begin{aligned} &\int_0^T e^{-i\omega_k t} f(t)dt = c_{k1}, \quad k = 1, 2, ..., \\ &\int_0^T e^{-i\omega_k t} u_1(t)dt = c_{k2}, \quad k = 1, 2, ..., \\ &\int_0^T e^{-i\omega_k t} u_2(t)dt = \frac{1}{b_k^1}\left(\delta_k - b_k c_{k1} - b_k^0 c_{k2}\right), \quad k = 1, 2, ... \end{aligned} \tag{30}$$

In (30), if we take $Q_{0k} = 0$, $Q_{1k} = 0$ and $Q_{1k}^{(1)} = 0$, we obtain the equivalent moment problem for the homogenous control problem given with the system (11)-(12).

3.3. **The structure of the general solution of the non-homogenous control problem for the system (11)-(12).** We prove the theorem determining the structure of the general solution of the control problem for linear systems.

Theorem 3.1. *If the functions $f_0(t)$, $u_{10}(t)$ and $u_{20}(t)$ form a special solution of the non-homogenous control problem of the system (11)-(12), then the general solution of this problem is found by*

$$\begin{aligned} f(t) &= f^*(t) + f_0(t), \\ u_1(t) &= u_1^*(t) + u_{10}(t), \\ u_2(t) &= u_2^*(t) + u_{20}(t), \end{aligned}$$

where the triplet $f^*(t)$, $u_1^*(t)$ and $u_2^*(t)$ is the general solution of the corresponding homogenous control problem.

Proof. According to our assumption the triplet $f_0(t)$, $u_{10}(t)$ and $u_{20}(t)$ is a solution of non-homogenous control problem

$$m\left(\frac{\partial}{\partial t}\right) Q_0 = \ell\left(x, \frac{\partial}{\partial x}\right) Q_0 + b(x) f_0(t),$$

$$R_1(Q_0) = u_{10}(t), \quad R_2(Q_0) = u_{20}(t),$$

$$Q_0(x,0) = Q_0(x), \qquad Q_0(x,T) = Q_1(x),$$
$$Q_0{}'_t(x,0) = Q^{(1)}(x), \quad Q_0{}'_t(x,T) = Q_1^{(1)}(x).$$

The triplet $f^*(t)$, $u_1^*(t)$ and $u_2^*(t)$ is a solution of the corresponding homogenous control problem. Now, suppose that the triplet $f(t)$, $u_1(t)$ and $u_2(t)$ is an arbitrary solution for the non-homogenous control problem for the system (11)-(12). Then the differences $f^*(t) = f(t) - f_0(t)$, $u_1^*(t) = u_1(t) - u_{10}(t)$ and $u_2^*(t) = u_2(t) - u_{20}(t)$ is clearly a solution of the homogenous control problem

$$m\left(\frac{\partial}{\partial t}\right) Q = \ell\left(x, \frac{\partial}{\partial x}\right) Q + b(x) f^*(t),$$

$$R_1(Q) = u_1^*(t), \quad R_2(Q) = u_2^*(t),$$

$$Q(x,0) = 0, \quad Q(x,T) = 0,$$
$$Q'_t(x,0) = 0, \quad Q'_t(x,T) = 0.$$

Thus, the proof is completed.

Example. Suppose that the control system is given by the equation

$$\frac{\partial^2 Q}{\partial t^2} = \frac{\partial^2 Q}{\partial x^2}, \quad 0 < x < 1, \quad 0 < t < T, \tag{31}$$

and boundary conditions

$$\begin{aligned} Q(0,t) &= u(t), \quad 0 \le t \le T, \\ Q(1,t) &= 0, \qquad 0 \le t \le T. \end{aligned} \tag{32}$$

We find the general solution of homogenous control problem for the system (31)-(32). If we perform Fourier transformation in the system (31)-(32), considering zero initial state conditions and zero final state conditions, we pass to the control system

$$\frac{d^2 \widetilde{Q}}{dx^2} + \omega^2 \widetilde{Q} = 0, \tag{33}$$

$$\widetilde{Q}(0,\omega) = \tilde{u}(\omega), \quad \widetilde{Q}(1,\omega) = 0, \tag{34}$$

dependent to the parameter. Then, for the system (33)-(34)

$$y_1(x,\omega) = \cos\omega x, \quad y_2(x,\omega) = \frac{\sin\omega x}{\omega},$$

$$R_1(y_1) = 1, \qquad R_1(y_2) = 0,$$
$$R_2(y_1) = \cos\omega, \quad R_2(y_2) = \frac{\sin\omega}{\omega}.$$

Thus,

$$\psi_1(x,\omega) = \frac{\sin\omega x}{\omega}, \quad \psi_2(x,\omega) = \frac{\sin\omega(1-x)}{\omega},$$
$$\Delta(\omega) = \frac{\sin\omega}{\omega}, \qquad \omega_k = k\pi, \quad k = 1,2,\dots\ .$$

The solution of the system (33)-(34) is found as

$$\widetilde{Q} = \frac{\sin\omega(1-x)\ \tilde{u}(\omega)}{\sin\omega}.$$

The interpolation condition for the function $\tilde{u}(\omega)$ is

$$\tilde{u}(\omega_k) = 0, \quad k = 1,2,\dots\ . \tag{35}$$

The solution of the interpolation problem (35) is found in [18] as

$$\tilde{u}(\omega) = \gamma(\omega) \sin \omega.$$

Here $\gamma(\omega)$ is a complete function such that $\tilde{u}(\omega) \in L^2(-\infty, \infty)$. For example, we can take

$$\gamma = \frac{c}{\omega},$$

where c is a constant. Then

$$\tilde{u}(\omega) = \frac{c \sin \omega}{\omega}.$$

Using the inverse Fourier transform

$$u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \tilde{u}(\omega) d\omega,$$

we find the equality

$$u(t) = \begin{cases} c, & 0 < t < 2, \\ 0, & t \in (-\infty, \infty) \setminus (0, 2). \end{cases}$$

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