ON IDEAL CONVERGENCE OF DOUBLE SEQUENCES IN THE TOPOLOGY INDUCED BY A FUZZY 2-NORM

LJUBIŠA D.R. KOČINAC1, MOHAMMAD H.M. RASHID2

Abstract. In this paper we introduce and investigate $I_2$-convergence, $I_2^*$-convergence, $I_2$-limit points, and $I_2$-cluster points of a double sequence in a fuzzy 2-normed linear space. We prove a decomposition theorem for $I_2$-convergence of double sequences. The notions of $I_2$-double Cauchy and $I_2^*$-double Cauchy sequence are defined, and some of their properties are studied.

Keywords: fuzzy number, fuzzy normed space, ideal convergence, ideal Cauchy sequence.

AMS Subject Classification: Primary 40A35; Secondary 03E72, 46A70, 54A20, 54A40.

1. Introduction

In 1965, Zadeh [41] introduced the notion of fuzzy sets and since then fuzzy set theory found very useful applications in various fields of mathematics and many other sciences. In particular, a number of papers deals with fuzzy real numbers introduced in [8]. In this paper we are interested in ideal convergence of double sequences in fuzzy 2-normed linear spaces.

The concept of 2-normed spaces was introduced by Gähler [20] in the 1960’s, and then this concept has been studied by many authors [7, 11, 12]; for more information see [37].

The idea of fuzzy norm was initiated by Katsaras [27], and Matloka [32] introduced convergence of sequences of fuzzy numbers. After that a big number of works dealing with fuzzy norms and fuzzy numbers, in particular with convergence of sequences of fuzzy numbers, appeared in the literature (see, for example, [15, 21, 22, 31, 35, 38]). By using fuzzy numbers Felbin [18] defined a fuzzy norm on a linear space whose associated fuzzy metric is of Kaleva and Seikkala type [24]. Cheng and Mordeson [9], and also Bag and Samanta [4] introduced a fuzzy norm on a linear space whose associated fuzzy metric is of Kramosil and Michalek type [30]. In [5], a comparative study of the fuzzy norms defined by Katsaras [30], Felbin [18], and Bag and Samanta [4] was given.

Using the concept of ideal, Kostyrko et al. [28] introduced the notion of ideal convergence which is a common generalization of ordinary convergence and statistical convergence [17, 19, 29, 40, 14] and provides a general framework for study of various kinds of convergence. Ideal and statistical convergence were studied in the fuzzy set theory context from different points of view (see [1, 2, 3, 6, 16, 23, 25, 26, 33, 36, 39]).

This paper is organized as follows: In the second section, we present some preliminary definitions and results related to fuzzy numbers, fuzzy normed spaces and ideal convergence. In the third section, we introduce the notions of $I_2^E$-convergence and $I_2^*E$-convergence of double sequences in a fuzzy 2-normed space $E$ and prove some basic results in this connection. We
also study the concepts of $I_2^F$-limit points and $I_2^F$-cluster points of double sequences in fuzzy 2-normed spaces. In fourth section, we introduce the notions of $I_2^F$-double Cauchy and $I_2^E$-double Cauchy sequences in a fuzzy 2-normed space.

Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural numbers and real numbers, respectively. $J$ denotes the closed unit interval $[0, 1] \subset \mathbb{R}$, and $I_2$ is an ideal on $\mathbb{N} \times \mathbb{N}$.

2. Definitions and Preliminaries

In this section we recall some basic definitions and notions related to 2-normed spaces, fuzzy numbers, fuzzy normed and fuzzy 2-normed spaces, and ideal convergence.

**Definition 2.1.** ([20]) Let $X$ be a real vector space of dimension $d$, $2 \leq d < \infty$. A 2-norm on $X$ is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies:

- (i) $\|x, y\| = 0$ if and only if $x$ and $y$ are linearly dependent;
- (ii) $\|x, y\| = \|y, x\|$ for all $x, y \in X$;
- (iii) $\|cx, y\| = |c|\|x, y\|$ for all $x, y \in X$ and $c \in \mathbb{R}$;
- (iv) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for all $x, y, z \in X$.

The pair $(X, \|\cdot,\cdot\|)$ is called a 2-normed space.

**Definition 2.2.** ([8], [18], [24]) A fuzzy real number, or simply fuzzy number, is a fuzzy set $X : \mathbb{R} \to [0, 1]$ having the following properties:

- (a) $X$ is normal (i.e. there exists a $t_0 \in \mathbb{R}$ such that $X(t_0) = 1$);
- (b) $X$ is fuzzy convex (i.e. for $r, s \in \mathbb{R}$ and $\lambda \in J = [0, 1]$, $X(\lambda r + (1-\lambda)s) \geq \min\{X(r), X(s)\}$);
- (c) $X$ is upper semi-continuous (i.e. $X^+([0,t+\varepsilon])$ is open in $\mathbb{R}$ for each $t \in J$ and each $\varepsilon > 0$);
- (d) The closure of the set $[X]_0 := \{t \in \mathbb{R} : X(t) > 0\}$ is compact.

Let $\mathcal{F}(\mathbb{R})$ be the set of all fuzzy real numbers. For $X \in \mathcal{F}(\mathbb{R})$, the $\alpha$-level set of $X$ [18] is defined as:

$$[X]_\alpha = \begin{cases} \{t \in \mathbb{R} : X(t) \geq \alpha\}, & \text{if } 0 < \alpha \leq 1; \\ \text{Cl}(\{t \in \mathbb{R} : X(t) > 0\}), & \text{if } \alpha = 0. \end{cases}$$

A real number $x$ can be considered as a fuzzy number $\overline{x}$ defined by

$$\overline{x}(t) = \begin{cases} 1, & \text{if } t = x; \\ 0, & \text{if } t \neq x, \end{cases}$$

i.e., $\mathbb{R}$ can be embedded in $\mathcal{F}(\mathbb{R})$.

It is easy to show that $X$ is a fuzzy number if and only if $[X]_\alpha$ is a nonempty bounded and closed interval for each $\alpha \in [0, 1]$. We denote this interval $[X]_\alpha = [X^-_\alpha, X^+_\alpha]$ (see [21]).

**Remark 2.3.** The above definition of fuzzy numbers slightly differs from that of [18], where $X^-_\alpha = -\infty$ and $X^+_\alpha = +\infty$ are also admissible, and the zero-level set is not considered.

A fuzzy number $X$ is called a non-negative fuzzy number if $X(t) = 0$ for $t < 0$. Let $\mathcal{F}^+(\mathbb{R})$ be the set of all non-negative fuzzy numbers. Clearly, $X \in \mathcal{F}^+(\mathbb{R})$ if and only if $X^-_\alpha \geq 0$ for each $\alpha \in J$, and $\overline{X} \in \mathcal{F}^+(\mathbb{R})$.

A partial order $\preceq$ on $\mathcal{F}(\mathbb{R})$ is defined by $X \preceq Y$ if and only if $X^-_\alpha \leq Y^-_\alpha$ and $X^+_\alpha \leq Y^+_\alpha$, for all $\alpha \in J$. The strict inequality $\prec$ on $\mathcal{F}(\mathbb{R})$ is defined by $X \prec Y$ if and only if $X^-_\alpha < Y^-_\alpha$ and $X^+_\alpha < Y^+_\alpha$, for all $\alpha \in J$.

Let $X, Y \in \mathcal{F}(\mathbb{R})$, define

$$\overline{d}(X, Y) = \sup_{\alpha \in [0, 1]} \max\{|X^-_\alpha - Y^-_\alpha|, |X^+_\alpha - Y^+_\alpha|\}.$$
Then $\overline{d}$ is called the supremum metric on $\mathcal{F}(\mathbb{R})$. It is known that $(\mathcal{F}(\mathbb{R}), \overline{d})$ is a complete metric space (for details see [24]). Let $(X_k)$ be a sequence in $\mathcal{F}(\mathbb{R})$ and $X_0 \in \mathcal{F}(\mathbb{R})$. We say that $(X_k)$ converges to $X_0$ with respect to the metric $\overline{d}$ if $\lim_{k \to \infty} \overline{d}(X_k, X_0) = 0$. In this case we write $X_k \overset{\overline{d}}{\to} X_0$ or $\overline{d} - \lim_{k \to \infty} X_k = X_0$.

Now we define the notion of fuzzy 2-normed space.

Let $E$ be a real vector space with the zero element $\theta$, let $\| \cdot \| : E \times E \to \mathcal{F}(\mathbb{R})$, and let the mappings $L, R : [0, 1] \times [0, 1] \to [0, 1]$ be symmetric, non-decreasing in both arguments and satisfy $L(0, 0) = 0$ and $R(1, 1) = 1$.

**Definition 2.4.** The quadruple $(E, \| \cdot \|, L, R)$ is called a fuzzy 2-normed space and $\| \cdot \|$ a fuzzy 2-norm, if the following axioms are satisfied:

- **(2FN1)** $\|X, Y\| = 0$ if and only if $X$ and $Y$ are linearly dependent;
- **(2FN2)** $\|\lambda X, Y\| = |\lambda| \|X, Y\|, \lambda \in \mathbb{R}$;
- **(2FN3)** For all $X, Y, Z \in E$,
  
  (i) $\|X + Y, Z\|(r + s) \geq L(\|X, Z\|(r), \|Y, Z\|(s)), \text{ whenever } r \leq \|X, Z\|_1, s \leq \|Y, Z\|_1$
  
  and $r + s \leq \|X + Y, Z\|_1$,

  (ii) $\|X + Y, Z\|(r + s) \geq R(\|X, Z\|(r), \|Y, Z\|(s)), \text{ whenever } r \geq \|X, Z\|_1, s \geq \|Y, Z\|_1$
  
  and $r + s \geq \|X + Y, Z\|_1$.

In the sequel we take $L(p, q) = \min\{p, q\}$ and $R(p, q) = \max\{p, q\}$, for all $p, q \in [0, 1]$ and write $(E, \| \cdot \|)$ or simply $E$, for such $L$ and $R$.

**Remark 2.5.** If $L = \min$, then the triangle inequality (2FN3)(i) in Definition 2.4 is equivalent to the triangle inequality $\|X + Y, Z\|_\alpha \leq \|X, Z\|_\alpha + \|Y, Z\|_\alpha$, for all $X, Y, Z \in E$ and $\alpha \in [0, 1]$, while the inequality (2FN3)(ii), with $R = \max$, is equivalent to $\|X + Y, Z\|_\alpha^+ \leq \|X, Z\|_\alpha^+ + \|Y, Z\|_\alpha^+$, for all $\alpha \in [0, 1]$ and $X, Y, Z \in E$.

In fact we have the following result.

**Lemma 2.6.** For $L = \min$ and $R = \max$, we have that for each $\alpha \in [0, 1]$, $\|X, Z\|_\alpha^-$ and $\|X, Z\|_\alpha^+$ are norms on $E$ in the usual sense.

The following example is similar to [21, Example 2.1] concerning fuzzy normed linear spaces.

**Example 2.1.** Let $(E, \| \cdot \|_u)$ be an ordinary 2-normed linear space. Then a fuzzy 2-norm on $E$ can be obtained as

1. $\|X, Y\| = 0$ if $X$ and $Y$ are linearly independent;

2. $\|X, Y\|(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq a\|X, Y\|_u \text{ or } t \geq b\|X, Y\|_u; \\ \frac{t}{(1-a)\|X, Y\|_u} - \frac{a}{1-a}, & \text{if } a\|X, Y\|_u \leq t \leq \|X, Y\|; \\ \frac{t}{(1-b)\|X, Y\|_u} - \frac{b}{1-b}, & \text{if } \|X, Y\| \leq t \leq b\|X, Y\|_u. \end{cases}$

if $X$ and $Y$ are linearly independent and $0 < a < 1$, $1 < b < \infty$.

Hence $(E, \| \cdot \|_u)$ is a fuzzy 2-normed space. The fuzzy 2-norm considered above is called a triangular fuzzy 2-norm.

For $X \in E$, $\varepsilon > 0$ and $\alpha \in [0, 1]$, the $(\varepsilon, \alpha)$-neighborhood of $X$ is the set

$U_X(\varepsilon, \alpha) = \{Y \in E : \|X - Y, Z\|_\alpha^+ < \varepsilon, \text{ for all } Z \in E\}$.

The $(\varepsilon, \alpha)$-neighborhood system at $X$ is the collection

$U_X = \{U_X(\varepsilon, \alpha) : \varepsilon > 0, \alpha \in [0, 1]\}$.
and the \((\varepsilon, \alpha)\)-neighborhood system for \(E\) is the union \(U = \bigcup_{X \in E} U_X\). It is easy to see that \(U\) generates a first countable Hausdorff topology on \(E\).

**Definition 2.7.** Let \((E, \|., .\|)\) be a fuzzy 2-norm space. A sequence \(\{X_k\}\) in \(E\) is said to be convergent to \(X_0 \in E\) with respect to the norm on \(E\), and we denote this by \(X_k \to X_0\), provided \(d - \lim_{k \to \infty} \|X_k - X_0, Z\| = \overline{0}\) for all \(Z \in E\), i.e., for every \(\varepsilon > 0\) there exists an integer \(k_0 = k_0(\varepsilon)\) in \(\mathbb{N}\) such that \(d(\|X_k - X_0, Z\|, \overline{0}) < \varepsilon\), for \(k \geq k_0\).

This is the same as to say that for every \(\varepsilon > 0\) there exists an integer \(k_0(\varepsilon)\) in \(\mathbb{N}\) such that \(\sup_{\alpha \in [0, 1]} \|X_k - X_0, Z\|_\alpha^+ = \|X_k - X_0, Z\|_0^+ < \varepsilon\), for \(k \geq k_0\).

In terms of neighborhoods, we have \(X_k \to X_0\), provided that for every \(\varepsilon > 0\) there exists an integer \(k_0(\varepsilon)\) in \(\mathbb{N}\) such that \(X_k \in U_{X_0}(\varepsilon, 0)\) for all \(k \geq k_0\) and all \(Z \in E\).

Finally, we give some basic facts about classic notions ideals and filters.

Let \(Y \neq \emptyset\). Then:

1. A family \(\mathcal{I}\) of subsets of \(Y\) is said to be an ideal in \(Y\) provided the following conditions hold: (i) if \(A, B \in \mathcal{I}\), then \(A \cup B \in \mathcal{I}\), and (ii) \(A \in \mathcal{I}\) and \(B \subset A\) imply \(B \in \mathcal{I}\). If \(Y \notin \mathcal{I}\), then \(\mathcal{I}\) is called a proper ideal.

2. A non-empty family \(\mathcal{F}\) of subsets of \(Y\) is said to be a filter on \(Y\) if (i) \(\emptyset \notin \mathcal{F}\), (ii) if \(A, B \in \mathcal{F}\), then \(A \cap B \in \mathcal{F}\), and (iii) \(A \in \mathcal{F}\) and \(A \subset B \in \mathcal{Y}\) imply \(B \in \mathcal{F}\).

A proper ideal \(\mathcal{I}\) is said to be admissible if \(\{x\} \in \mathcal{I}\) for each \(x \in Y\). An admissible ideal \(\mathcal{I}\) on \(\mathbb{N}\) is said to have the property (AP) \([28]\) if for any sequence \(\{A_1, A_2, \ldots\}\) of pairwise disjoint sets of \(\mathcal{I}\), there is a sequence \(\{B_1, B_2, \ldots\}\) of sets such each symmetric difference \(A_i \Delta B_i (i = 1, 2, \ldots)\) is finite and \(\bigcup_{i=1}^{\infty} B_i \in \mathcal{I}\).

If \(\mathcal{I}\) is a proper ideal on \(Y\), then the family \(\mathcal{F}(\mathcal{I})\) is a filter in \(Y\). It is called the filter associated with the ideal \(\mathcal{I}\).

In what follows the symbol \(\mathcal{I}_2\) denotes an ideal on \(\mathbb{N} \times \mathbb{N}\), and \((E, \|., .\|)\) is a fuzzy 2-normed space.

**3. IDEAL CONVERGENCE IN FUZZY 2-NORMED LINEAR SPACES**

In this section we introduce the notions of \(\mathcal{I}_2^F\)-convergence and \(\mathcal{I}_2^E\)-convergence of a double sequence in a fuzzy 2-normed space \((E, \|., .\|)\) and present some basic results on this convergence. We also introduce the notions of \(\mathcal{I}_2\)-limit point and \(\mathcal{I}_2\)-cluster point of a double sequence in \((E, \|., .\|)\).

We begin with the following definition.

**Definition 3.1.** A double sequence \(\{X_{jk}\}\) in a fuzzy 2-normed space \((E, \|., .\|)\) is said to be \(E\)-convergent to \(X_0\) if for every \(\varepsilon > 0\) and each \(Z \in E\) there exists a positive integer \(n_0 = n_0(\varepsilon)\) such that \(X_{jk}, Z \in U_{X_0}(\varepsilon, 0)\) for each \(j, k \geq n_0\).

In this case we write \(E\lim \|X_{jk} - X_0, Z\|_0^+ = 0\).

**Definition 3.2.** Let \((E, \|., .\|)\) be a fuzzy 2-normed space and \(\mathcal{I}_2\) an ideal on \(\mathbb{N} \times \mathbb{N}\). A double sequence \(\{X_{jk}\}\) in \(E\) is said to be \(\mathcal{I}_2^E\)-convergent to \(X_0 \in E\) with respect to the fuzzy 2-norm on \(E\) if for each \(\varepsilon > 0\) and each \(Z \in E\), the set \(A(\varepsilon) := \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ \geq \varepsilon\}\) belongs to \(\mathcal{I}_2\).
In this case, we write \( X_{jk} \xrightarrow{TE} X_0 \). The element \( X_0 \) is called the \( I^E_2 \)-limit of \( \{X_{jk}\} \) in \( E \).

**Remark 3.3.** (a) In terms of neighborhoods, we have \( X_{jk} \xrightarrow{2E} X_0 \), provided that for each \( \varepsilon > 0 \) and \( Z \in E \),

\[
\{(j, k) \in \mathbb{N} \times \mathbb{N} : X_{jk}, Z \notin U_{X_0}(\varepsilon, 0)\} \in I_2.
\]

The above definition can be expressed also in the following way:

\[
X_{jk} \xrightarrow{2E} X_0 \iff I^E_2 \leftarrow \lim_{j,k \to \infty} \|X_{jk} - X_0, Z\|^+_0 = 0, \text{ for all } Z \in E.
\]

(b) Note that \( I^E_2 - \lim_{j,k \to \infty} \|X_{jk} - X_0, Z\|^+_0 = 0, \text{ for all } Z \in E \) implies

\[
I^E_2 - \lim \|X_{jk} - X_0, Z\|_\alpha^+ = I^E_2 - \lim \|X_{jk} - X_0, Z\|_\alpha^+
\]
for each \( \alpha \in [0, 1] \) and each \( Z \in E \).

(It is because \( 0 \leq \|X_{jk} - X_0, Z\|_\alpha^- \leq \lim \|X_{jk} - X_0, Z\|_\alpha^+ \leq \|X_{jk} - X_0, Z\|_0^+ \), holds for each \( (j, k) \in \mathbb{N} \times \mathbb{N} \), \( \alpha \in [0, 1] \) and each \( Z \in E \).)

**Example 3.1.** (1) If we take \( I_2 = I_{fin} = \{A \subseteq \mathbb{N} \times \mathbb{N} : A \text{ is finite}\} \), then \( I_{fin} \) is a non-trivial admissible ideal of \( \mathbb{N} \times \mathbb{N} \), and the corresponding convergence coincides with ordinary convergence with respect to the fuzzy 2-norm on \( E \) (Definition 3.1).

(2) If we take \( I_2 = I_{\delta_1} = \{A \subseteq \mathbb{N} \times \mathbb{N} : \delta_1(A) = 0\} \), then \( I_{\delta_1} \) is a non-trivial admissible ideal of \( \mathbb{N} \times \mathbb{N} \), and the corresponding convergence coincides with statistical convergence with respect to the fuzzy 2-norm on \( E \).

**Proposition 3.4.** Let \( (E, ||\cdot||) \) be a fuzzy 2-normed space. If a double sequence \( \{X_{jk}\} \) is \( I^E_2 \)-convergent with respect to the norm on \( E \), then \( I^E_2 \)-limit is unique.

**Proof.** Let us assume that \( X_{jk} \xrightarrow{IE} X_0 \) and \( X_{jk} \xrightarrow{IE} Y_0 \), where \( X_0 \neq Y_0 \). Since \( X_0 \neq Y_0 \), select \( \varepsilon > 0 \) so that \( U_{X_0}(\varepsilon, 0) \) and \( U_{Y_0}(\varepsilon, 0) \) are disjoint neighborhoods of \( X_0 \) and \( Y_0 \). Since \( X_0 \) and \( Y_0 \) both are \( I^E_2 \)-limit of the sequence \( \{X_{jk}\} \), we have that for each \( Z \in E \) the sets

\[
A = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, Z\|^+_0 \geq \varepsilon\}
\]

and

\[
B = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - Y_0, Z\|^+_0 \geq \varepsilon\}
\]

both belong to \( I_2 \). This implies that the sets

\[
A^c = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, Z\|^+_0 < \varepsilon\}
\]

and

\[
B^c = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - Y_0, Z\|^+_0 < \varepsilon\}
\]

belong to \( \mathcal{F}(I_2) \). In this way we obtain a contradiction to the fact that the neighborhoods \( U_{X_0}(\varepsilon, 0) \) and \( U_{Y_0}(\varepsilon, 0) \) of \( X_0 \) and \( Y_0 \) are disjoint. Hence we have \( X_0 = Y_0 \). \( \square \)

**Proposition 3.5.** Let \( (E, ||\cdot||) \) be a fuzzy 2-normed space. Then we have

1. If \( E \)-lim \( \|X_{jk} - X_0, Z\|^+_0 = 0 \), then \( I^E_2 \)-lim \( \|X_{jk} - X_0, Z\|^+_0 = 0 \);
2. If \( X_{jk} \xrightarrow{IE} X_0 \) and \( Y_{jk} \xrightarrow{IE} Y_0 \), then \( X_{jk} + Y_{jk} \xrightarrow{IE} X_0 + Y_0 \);
3. If \( X_{jk} \xrightarrow{IE} X_0 \) and \( c \in \mathbb{R} \), then \( cX_{jk} \xrightarrow{IE} cX_0 \);
4. If \( X_{jk} \xrightarrow{IE} X_0 \) and \( Y_{jk} \xrightarrow{IE} Y_0 \), then \( X_{jk} \cdot Y_{jk} \xrightarrow{IE} X_0 \cdot Y_0 \);
(5) If \( X_{jk} \preceq Y_{jk} \preceq Z_{jk} \) for all \((j,k) \in \mathbb{N} \times \mathbb{N}\) belonging to the set \( B \in \mathcal{F}(I_2)\), and \( X_{jk} \overset{\mathcal{F}}{\longrightarrow} X_0 \) and \( Z_{jk} \overset{\mathcal{F}}{\longrightarrow} Z_0 \), then \( Y_{jk} \overset{\mathcal{F}}{\longrightarrow} Y_0 \).

**Proof.** (1) Suppose that \( E \)-\(\lim\) \( \|X_{jk} - X_0, Z\|_0^+ = 0\). Let \( \varepsilon > 0 \) and \( Z \in E \) any nonzero element. Then there exists a positive integer \( n \) such that \( \|X_{jk} - X_0, Z\|_0^+ < \varepsilon \) for each \( j, k \geq n \). Since

\[ A = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ \geq \varepsilon\} \subseteq \{1, 2, \ldots, n-1\} \times \{1, 2, \ldots, n-1\} \]

and the ideal \( I_2 \) is admissible, we have \( A \in I_2 \). This shows that \( \mathcal{F}_E \)-\(\lim\) \( \|X_{jk} - X_0, Z\|_0^+ = 0 \).

(2) Suppose that \( X_{jk} \overset{\mathcal{F}}{\longrightarrow} X_0 \) and \( Y_{jk} \overset{\mathcal{F}}{\longrightarrow} Y_0 \). Since \( \|\cdot\|_0^+ \) is a 2-norm in the usual sense, we get

\[ \|(X_{jk} + Y_{jk}) - (X_0 + Y_0), Z\|_0^+ \leq \|X_{jk} - X_0, Z\|_0^+ + \|Y_{jk} - Y_0, Z\|_0^+ \]

for all \((j, k) \in \mathbb{N} \times \mathbb{N}\). Put

\[ A(\varepsilon) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|(X_{jk} + Y_{jk}) - (X_0 + Y_0), Z\|_0^+ \geq \varepsilon\} \]

\[ A_1(\frac{\varepsilon}{2}) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ \geq \varepsilon\} \]

\[ A_2(\frac{\varepsilon}{2}) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|Y_{jk} - Y_0, Z\|_0^+ \geq \varepsilon\} \]

By assumption, we have that \( A_1(\frac{\varepsilon}{2}) \) and \( A_2(\frac{\varepsilon}{2}) \) belong to \( I_2 \), and so \( A_1(\frac{\varepsilon}{2}) \cup A_2(\frac{\varepsilon}{2}) \in I_2 \). From (1) it follows that \( A(\varepsilon) \subseteq A_1(\frac{\varepsilon}{2}) \cup A_2(\frac{\varepsilon}{2}) \). This implies that \( A(\varepsilon) \in I_2 \). This proves (2).

(3) Since \( X_{jk} \overset{\mathcal{F}}{\longrightarrow} X_0 \), we have

\[ A(1) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ < 1\} \in \mathcal{F}(I_2) \]

Now \( \|\cdot\|_0^+ \) is a 2-norm in the usual sense, so that

\[ \|X_{jk} Y_{jk} - X_0 Y_0, Z\|_0^+ \leq \|X_{jk}, Z\|_0^+ \|Y_{jk} - Y_0, Z\|_0^+ + \|Y_0, Z\|_0^+ \|X_{jk} - X_0, Z\|_0^+ \]

For \((j, k) \in A(1)\), we have \( \|X_{jk}, Z\|_0^+ \leq \|X_0, Z\|_0^+ + 1 \) and it follows that

\[ \|X_{jk} Y_{jk} - X_0 Y_0, Z\|_0^+ \leq (\|X_0, Z\|_0^+ + 1) \|Y_{jk} - Y_0, Z\|_0^+ + \|Y_0, Z\|_0^+ \|X_{jk} - X_0, Z\|_0^+ \]

(2) Let \( \varepsilon > 0 \) be given. Choose \( \lambda > 0 \) such that

\[ 0 < 2\lambda < \frac{\varepsilon}{\|Y_0, Z\|_0^+ + \|X_0, Z\|_0^+ + 1} \]

Since \( X_{jk} \overset{\mathcal{F}}{\longrightarrow} X_0 \) and \( Y_{jk} \overset{\mathcal{F}}{\longrightarrow} Y_0 \), the sets

\[ A_1(\lambda) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ < \lambda\} \]

and

\[ A_2(\lambda) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|Y_{jk} - Y_0, Z\|_0^+ < \lambda\} \]

belong to \( \mathcal{F}(I_2) \).

Obviously, \( A(1) \cap A_1(\lambda) \cap A_2(\lambda) \in \mathcal{F}(I_2) \) and for each \((j, k) \in A(1) \cap A_1(\lambda) \cap A_2(\lambda)\), we have from (2) and (3),

\[ \|X_{jk} Y_{jk} - X_0 Y_0\|_0^+ < \varepsilon \]

This implies that \( \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, Y_0\|_0^+ \geq \varepsilon\} \in I_2 \), i.e., \( X_{jk} \cdot Y_{jk} \overset{\mathcal{F}}{\longrightarrow} X_0 \cdot Y_0 \).

(4) Let \( c \in \mathbb{R} \). If \( c \neq 0 \), we have nothing to prove, so we assume that \( c \neq 0 \). Let \( \varepsilon > 0 \) be given. Since \( \|\cdot\|_0^+ \) is a 2-norm in usual sense, \( \|cX_{jk}, Z\|_0^+ = |c|\|X_{jk}, Z\|_0^+ \).

Since \( X_{jk} \overset{\mathcal{F}}{\longrightarrow} X_0 \), we have

\[ A(\varepsilon) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ \geq \varepsilon\} \in I_2 \].
Let $A_1(\varepsilon) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|cX_{jk} - cX_0, Z\|_0^+ \geq \varepsilon\}$. We need to show that $A_1(\varepsilon)$ is contained in $A(\varepsilon_1)$. Let $(t, s) \in A_1(\varepsilon)$, then $\varepsilon \leq \|cX_{ts} - cX_0\|_0^+ = |c|\|X_{ts} - X_0\|_0^+$. This implies that $\|X_{ts} - X_0\|_0 \geq \varepsilon_1$. Therefore $(t, s) \in A(\varepsilon_1)$. Then we have $A_1(\varepsilon) \subset A(\varepsilon_1)$. By the definition of the ideal, we get $A_1(\varepsilon) \in \mathcal{I}_2$ which proves (4).

(5) Let $\varepsilon > 0$ and $W \in E$ be given. From $X_{jk} \overset{\mathcal{F}}{\to} X_0$ it follows

$$A_1(\varepsilon) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, W\|_0^+ \geq \varepsilon\} \in \mathcal{I}_2,$$

and from $Z_{jk} \overset{\mathcal{F}}{\to} X_0$ it follows

$$A_2(\varepsilon) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|Z_{jk} - X_0, W\|_0^+ \geq \varepsilon\} \in \mathcal{I}_2.$$

We shall prove

$$C := \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|Y_{jk} - X_0, W\|_0^+ \geq \varepsilon\} \subset A_1(\varepsilon) \cup A_2(\varepsilon) \cup (\mathbb{N}^2 \setminus B).$$

Let $(p, q) \in C$. If $(p, q) \in \mathbb{N}^2 \setminus B$, then $(p, q) \in A_1(\varepsilon) \cup A_2(\varepsilon) \cup (\mathbb{N}^2 \setminus B)$. Assume now $(p, q) \in B$. Then $\|Y_{pq} - X_0, W\|_0^+ \geq \varepsilon$. Since $Z_{pq} \geq Y_{pq}$ we have $\|Z_{pq} - X_0, W\|_0^+ \geq \varepsilon$, hence $(p, q) \in A_2(\varepsilon)$. Therefore, $(p, q) \in A_1(\varepsilon) \cup A_2(\varepsilon) \cup (\mathbb{N}^2 \setminus B)$. Because the last set is in $\mathcal{I}_2$, we get $C \in \mathcal{I}_2$, i.e.

$Y_{jk} \overset{\mathcal{F}}{\to} X_0$.

Lemma 3.6. Let $\mathcal{I}_2$ be an admissible ideal with the property (AP). If $\{P_j\}_{j=1}^\infty$ is a countable collection of subsets of $\mathbb{N} \times \mathbb{N}$ such that $P_j \in \mathcal{F}(\mathcal{I}_2)$ for each $j$, then there exists a set $P \subset \mathbb{N} \times \mathbb{N}$ such that $P \setminus P_j$ is finite for all $j$.

Proof. Let $A_1 = \mathbb{N}^2 \setminus P_1$, $A_m = (\mathbb{N}^2 \setminus P_m) \setminus (A_1 \cup A_2 \cdots A_{m-1})$, $m = 2, 3, \ldots$. Evidently, $A_i \in \mathcal{I}_2$ for each $i$, and $A_i \cap A_j = \emptyset$ when $i \neq j$. Then, by property (AP) of $\mathcal{I}_2$, we conclude that there exists a countable family of sets $\{B_1, B_2, \cdots\}$ such that $A_j \Delta B_j$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^\infty B_j \in \mathcal{I}_2$. Put $P = \mathbb{N}^2 \setminus B$. It is clear that $P \in \mathcal{F}(\mathcal{I}_2)$.

Now we prove that the set $P \setminus P_j$ is finite for each $i$. Let $j_0 \in \mathbb{N}$ be given. Since each $A_j \Delta B_j$ ($j = 1, \cdots, j_0$) is a finite set, there exists $(n_0, m_0) \in \mathbb{N} \times \mathbb{N}$ such that

$$\bigcup_{j=1}^{j_0} B_j \cap \{(n, m) \in \mathbb{N}^2 : n > n_0, m > m_0\} = \bigcup_{j=1}^{j_0} A_j \cap \{(n, m) \in \mathbb{N}^2 : n > n_0, m > m_0\}. \quad (4)$$

If $n > n_0, m > m_0$ and $(n, m) \notin B$, then $(n, m) \notin \bigcup_{j=1}^{j_0} B_j$ and, by (4), $(n, m) \notin \bigcup_{j=1}^{j_0} A_j$. Since $A_{j_0} = (\mathbb{N}^2 \setminus P_{j_0}) \setminus \bigcup_{j=1}^{j_0-1} A_j$ and $(n, m) \notin A_{j_0}$, we have $(n, m) \notin \bigcup_{j=1}^{j_0-1} A_j$, and thus $(n, m) \in P_{j_0}$ for all $n$ and $m$ with $n > n_0, m > m_0$. Therefore, we get $(n, m) \in P$ and $(n, m) \in P_{j_0}$ for all $(n, m) \in \mathbb{N}^2$ with $n > n_0, m > m_0$. This shows that the set $P \setminus P_{j_0}$ is finite and the lemma is proved.

Theorem 3.7. Let $\mathcal{I}_2$ be an admissible ideal with the property (AP). Let $(E, \|\|, \|\|)$ be a fuzzy 2-normed space and $\{X_{jk}\}$ be a double sequence in $E$. Then $\{X_{jk}\}$ is an $\mathcal{I}_2^\mathcal{F}$-convergent sequence in $E$ if and only if there is an $E$-convergent double sequence $\{Y_{jk}\}$ such that $\{(j, k) \in \mathbb{N} \times \mathbb{N} : X_{jk} \neq Y_{jk}\} \in \mathcal{I}_2$. 

Proof. Suppose $X_{jk} \xrightarrow{\mathcal{I}^E} X_0$. For each $n \in \mathbb{N}$ and a non-zero $Z \in E$, let

$$A_n = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ < \frac{1}{n}\}.$$ 

Then $A_n \in \mathcal{F}(\mathcal{I}_2)$ for each $n \in \mathbb{N}$.

Since $\mathcal{I}_2$ is admissible ideal with the property $(AP)$, by Lemma 3.6 there exists $A \subset \mathbb{N} \times \mathbb{N}$ such that $A \in \mathcal{F}(\mathcal{I})$ and the set $\mathcal{A} \setminus A_n$ is finite for each $n$. Observe that $X_{jk} \xrightarrow{(A)} X_0$, i.e., for each $\varepsilon > 0$, there exists an integer $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $j, k \geq n_0$ and $(j,k) \in A$ implies $\|X_{jk} - X_0, Z\|_0^+ < \varepsilon$.

Define a sequence $\{Y_{jk}\}$ in $E$ as

$$Y_{jk} = \begin{cases} X_{jk}, & \text{for } (j,k) \in A; \\ X_0, & \text{for } (j,k) \in (\mathbb{N} \times \mathbb{N}) \setminus A. \end{cases}$$

The sequence $\{Y_{jk}\}$ is $E$-convergent to $X_0$ with respect to the fuzzy norm on $E$. Thus we have $\{(j,k) \in \mathbb{N} \times \mathbb{N} : X_{jk} \neq Y_{jk}\} \in \mathcal{I}_2$.

Next suppose that $\{(j,k) \in \mathbb{N} \times \mathbb{N} : X_{jk} \neq Y_{jk}\} \in \mathcal{I}_2$ and $Y_{jk} \rightarrow X_0$. Let $\varepsilon > 0$ be given. Then for each $n$ and a non-zero $Z \in E$, we can write

$$\{j, k \leq n : \|X_{jk} - X_0, Z\|_0^+ \geq \varepsilon\} \subset \{j, k \leq n : X_{jk} \neq Y_{jk}\} \cup \{j, k \leq n : \|X_{jk} - X_0, Z\|_0^+ > \varepsilon\}.$$ 

(5)

Since first set on the right side of $(5)$ belongs to $\mathcal{I}_2$, and the second set contain in a fixed number of integers and thus belongs to $\mathcal{I}_2$, we conclude that $\{(j,k) : j, k \leq n, \|X_{jk} - X_0, Z\|_0^+ \geq \varepsilon\}$ belongs to $\mathcal{I}_2$. This achieves the proof. $\square$

Now we prove a decomposition theorem for $\mathcal{I}_2^E$-convergent sequences.

**Theorem 3.8.** Let $\{X_{jk}\}$ be a double sequence in a fuzzy 2-normed space $(E, \|\cdot,\|)$ and $\mathcal{I}_2$ be an admissible ideal. If there exist two sequences $\{Y_{jk}\}$ and $\{Z_{jk}\}$ in $E$ such that $X_{jk} = Y_{jk} + Z_{jk}$; $Y_{jk}$ $E$-converges to $X_0$ and $\text{supp}(Z_{jk}) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : Z_{jk} \neq 0\} \in \mathcal{I}_2$, then $X_{jk} \xrightarrow{\mathcal{I}_2^E} X_0$.

**Proof.** Let $\{Y_{jk}\}$ and $\{Z_{jk}\}$ be double sequences in $E$ as in the statement of the theorem and $H = \text{supp}(Z_{jk})$. Let $\varepsilon > 0$ and $W \in E$ be given. Since $A_1 = \{(j,k) \in \mathbb{N}^2 : \|Z_{jk} - \overline{0}, W\|_0^+ \geq \varepsilon/2\} \subset \text{supp}(Z_{jk}) = H$, we have $A_1 \in \mathcal{I}_2$. Further,

$$\|X_{jk} - X_0, W\|_0^+ = \|Y_{jk} + Z_{jk} - \overline{0} - X_0, W\|_0^+ \leq \|Y_{jk} - X_0, W\|_0^+ + \|Z_{jk} - \overline{0}, W\|_0^+$$

implies

$$\{(j,k) \in \mathbb{N}^2 : \|X_{jk} - X_0, W\|_0^+ < \varepsilon\} \supset \{(j,k) \in \mathbb{N}^2 : \|Y_{jk} - X_0, W\|_0^+ < \varepsilon/2\}$$

$$\cap \{(j,k) \in \mathbb{N}^2 : \|Z_{jk} - \overline{0}\|_0^+ < \varepsilon/2\}.$$ 

The sets on the right side are both in $\mathcal{F}(\mathcal{I}_2)$, so that the set on the left side is also in $\mathcal{F}(\mathcal{I}_2)$. Therefore, $\{(j,k) \in \mathbb{N}^2 : \|X_{jk} - X_0, W\|_0^+ \geq \varepsilon\} \in \mathcal{I}_2$, i.e. $X_{jk} \xrightarrow{\mathcal{I}_2^E} X_0$. $\square$

**Definition 3.9.** Let $(E, \|\cdot,\|)$ be a fuzzy 2-normed space. We say that a double sequence $\{X_{jk}\}$ in $E$ is $\mathcal{I}_2^E$-convergent to $X_0 \in E$ with respect to the 2-norm on $E$ if there exists a subset

$$K = \{(j_m, k_m) : j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots \} \subset \mathbb{N} \times \mathbb{N}$$

such that $K \in \mathcal{F}(\mathcal{I}_2)$ and $E\text{-}\lim_{m \rightarrow \infty} \|X_{j_m k_m} - X_0, Z\| = 0$ for each non-zero $Z \in E$.

In this case we write $X_{jk} \xrightarrow{\mathcal{I}_2^E} X_0$. 

**Theorem 3.10.** Let \((E, \|\cdot\|)\) be a fuzzy 2-normed space and \(\mathcal{I}_2\) be an admissible ideal. If \(X_{jk} \overset{\mathcal{I}_2}{\to} X_0\), then \(X_{jk} \overset{\mathcal{I}_E}{\to} X_0\).

**Proof.** Suppose that \(X_{jk} \overset{\mathcal{I}_E}{\to} X_0\). Then by definition, there exists

\[
K = \{(j_m, k_m) \in \mathbb{N} \times \mathbb{N} : j_1 < j_2 < \cdots : k_1 < k_2 < \cdots\} \subseteq \mathcal{F}(\mathcal{I}_2)
\]

such that for every \(m \in \mathbb{N}\), there exists \(n_0 \in \mathbb{N}\) such that \(\|X_{j_m, k_m} - X_0, Z\|_0^+ < \varepsilon\) for every \(m \geq n_0\). Since

\[
X = \{\|X_{j_m, k_m} - X_0, Z\|_0^+ \geq \varepsilon\}
\]

is contained in

\[
B = \{j_1, j_2, \cdots, j_{n_0-1}, k_1, k_2, \cdots, k_{n_0-1}\}
\]

and the ideal \(\mathcal{I}_2\) is admissible, we have \(A \subseteq \mathcal{I}_2\). Hence

\[
\{j, k\} \in \mathbb{N} \times \mathbb{N} : \|X_{j_m, k_m} - X_0, Z\|_0^+ \geq \varepsilon\} \subseteq K \cup B \in \mathcal{I}_2
\]

for \(\varepsilon > 0\) and nonzero \(Z \in E\). Therefore, we conclude that

\[
X_{jk} \overset{\mathcal{I}_F}{\to} X_0.
\]

\[
\square
\]

**Theorem 3.11.** Let \(\mathcal{I}_2\) be an admissible ideal with the property (AP) and \((E, \|\cdot\|)\) be fuzzy 2-normed space and \(\{X_{jk}\}\) be a double sequence in \(E\). Then \(X_{jk} \overset{\mathcal{I}_E}{\to} X_0\) implies \(X_{jk} \overset{\mathcal{I}_2}{\to} X_0\).

**Proof.** Let \(X_{jk} \overset{\mathcal{I}_E}{\to} X_0\). Then by definition, for every \(\varepsilon > 0\) and a non-zero \(Z \in E\), there exists an integer \(n = n(\varepsilon)\) such that the set

\[
B(\varepsilon) = \{j, k\} \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ \geq \varepsilon\} \subseteq \mathcal{I}_2
\]

For \(m \in \mathbb{N}\), we define the set \(P_m\) as follows:

\[
P_1 = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ \geq 1\}
\]

and

\[
P_m = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \frac{1}{m} \leq \|X_{jk} - X_0, Z\|_0^+ < \frac{1}{m-1}\}, \text{ for } m \geq 2 \in \mathbb{N}.
\]

It is clear that \(P_1, P_2, \cdots\) is a countable family of mutually disjoint sets belonging to \(\mathcal{I}_2\), then by the property (AP) of \(\mathcal{I}_2\), there is a countable family of sets \(\{Q_1, Q_2, \cdots\}\) in \(\mathcal{I}_2\) such that \(P_j \Delta Q_j\) is a finite set for each \(j \in \mathbb{N}\) and \(Q = \bigcup_{j=1}^{\infty} Q_j \in \mathcal{I}_2\). Since \(Q \in \mathcal{I}_2\), so there a set \(B = \mathbb{N} \setminus Q\). To prove the result it is sufficient to show that \(X_{jk} \overset{(B)}{\to} X_0\). Let \(\xi > 0\) be given. Choose an integer \(p\) such that \(\xi > \frac{1}{p+1}\). Thus, we have

\[
\{(j, k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ \geq \xi\} \subseteq \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_0, Z\|_0^+ \geq \frac{1}{p+1}\}
\]

\[
= \bigcup_{m=1}^{p+1} P_m.
\]
Since $P_m \cap Q_m$ is a finite set for each $m = 1, \cdots, p + 1$, there exists $(j_0, k_0) \in \mathbb{N} \times \mathbb{N}$ such that 
\[
\left( \bigcup_{m=1}^{p+1} Q_m \right) \cap \{(j, k) \in \mathbb{N} \times \mathbb{N}: j \geq j_0, k \geq k_0 \} = \bigcup_{m=1}^{p+1} P_m \cap \{(j, k) \in \mathbb{N} \times \mathbb{N}: j \geq j_0, k \geq k_0 \}.
\]

If $j \geq j_0$ and $k \geq k_0$ and $(j, k) \in Q$. This implies that $(j, k) \notin \bigcup_{m=1}^{p+1} Q_m$ and so $(j, k) \notin \bigcup_{m=1}^{p+1} P_m$. Thus for every $j \geq j_0$ and $k \geq k_0$ and $(j, k) \in B$, from (6), we get $\|X_{jk} - X_0, Z\|_0^+ < \xi$. This shows $X_{jk} \rightarrow_{(B)} X_0$. This completes the proof. □

3.1. $I_2$-limit points and $I_2$-cluster points. In this subsection we introduce and consider the notions of $I_2$-limit points and $I_2$-cluster points of sequences in a fuzzy 2-normed space.

**Definition 3.12.** Let $\{X_{jk}\}$ be a double sequence in a fuzzy 2-normed space $(E, \|., .\|)$ and $I_2$ an ideal on $\mathbb{N} \times \mathbb{N}$. Then:

1. an element $W \in E$ is said to be an $I_2$-limit point of $\{X_{jk}\}$ provided that there is a set $K = \{(j_1, k_1), j_1 < j_2 < \cdots; k_1 < k_2 < \cdots\} \subset \mathbb{N} \times \mathbb{N}$ such that $K \notin I_2$ and $\lim_{m} \|X_{jmk_m} - W, Z\|_0^+ = 0$ for each a non-zero $Z \in E$;

2. an element $Y \in E$ is said to be an $I_2$-cluster point of $X_{jk}$ if for each $\varepsilon > 0$ and a non-zero $Z \in E$, the set $\{(j, k) \in \mathbb{N} \times \mathbb{N}: \|X_{jk} - Y, Z\|_0^+ < \varepsilon\} \notin I_2$.

We denote by $L^E_{I_2}(X_{jk})$ and $C^E_{I_2}(X_{jk})$ the set of all $I_2$-limit points and $I_2$-cluster points of a sequence $\{X_{jk}\}$ in $(E, \|., .\|)$.

**Theorem 3.13.** Let $I_2$ be an admissible ideal on $\mathbb{N} \times \mathbb{N}$. Then for any sequence $\{X_{jk}\}$ in a fuzzy 2-normed space $(E, \|., .\|)$, we have $L^E_{I_2}(X_{jk}) \subset C^E_{I_2}(X_{jk})$.

**Proof.** Assume that $W \in L^E_{I_2}(X_{jk})$. Then by definition there is a set $K = \{(j_1, k_1), j_1 < j_2 < \cdots; k_1 < k_2 < \cdots\} \subset \mathbb{N} \times \mathbb{N}$ such that $K \notin I_2$ and
\[
\lim_{m} \|X_{jmk_m} - W, Z\|_0^+ = 0 \quad \text{for each non-zero } Z \in E. \tag{7}
\]

Let $\varepsilon > 0$ and non-zero $Z \in E$ be given. According to (7), there exists an integer $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that for each $m \geq n_0$, we get $\|X_{jmk_m} - W, Z\|_0^+ < \varepsilon$.

Thus, we have
\[
K \setminus \{(j_1, k_1), \cdots, (j_{n_0}, k_{n_0})\} \subset \{(j, k) \in \mathbb{N} \times \mathbb{N}: \|X_{jmk_m} - W, Z\|_0^+ < \varepsilon\}.
\]

This implies that $\{(j, k) \in \mathbb{N} \times \mathbb{N}: \|X_{jmk_m} - W, Z\|_0^+ < \varepsilon\} \notin I_2$. Hence $W \in C^E_{I_2}(X_{jk})$. □

**Theorem 3.14.** Let $\{X_{jk}\}$ be a double sequence in a fuzzy 2-normed space $(E, \|., .\|)$. If $X_{jk} \rightarrow_{(B)} X_0$, then $L^E_{I_2}(X_{jk}) = C^E_{I_2}(X_{jk}) = \{X_0\}$.

**Proof.** Assume that $X_{jk} \rightarrow_{(B)} X_0$. Then for each $\varepsilon > 0$ and a non-zero $Z \in E$, the set
\[
\{(j, k) \in \mathbb{N} \times \mathbb{N}: \|X_{jk} - X_0, Z\|_0^+ \geq \varepsilon\} \in I_2,
\]

that is
\[
\{(j, k) \in \mathbb{N} \times \mathbb{N}: \|X_{jk} - X_0, Z\|_0^+ < \varepsilon\} \notin I_2,
\]

which implies that $X_0 \in C^E_{I_2}(X_{jk})$. 


We assume that there exists at least one \( Y_0 \in C_{\mathcal{T}_2}^E(X_{jk}) \) such that \( Y_0 \neq X_0 \). Then there exists \( \epsilon > 0 \) such that

\[
\{(j, k) \in \mathbb{N} \times \mathbb{N} : ||X_{jk} - Y_0, Z||_0^+ < \epsilon \} \subset \{(j, k) \in \mathbb{N} \times \mathbb{N} : ||X_{jk} - X_0, Z||_0^+ \geq \epsilon \}
\]

holds for each non-zero \( Z \in E \). But \( \{(j, k) \in \mathbb{N} \times \mathbb{N} : ||X_{jk} - X_0, Z||_0^+ \geq \epsilon \} \in \mathcal{T}_2 \) implies that \( \{(j, k) \in \mathbb{N} \times \mathbb{N} : ||X_{jk} - X_0, Z||_0^+ < \epsilon \} \in \mathcal{T}_2 \), which contradicts that \( Y_0 \in C_{\mathcal{T}_2}^E(X_{jk}) \). Thus we have \( C_{\mathcal{T}_2}^E(X_{jk}) = \{X_0\} \).

On the other hand, from \( X_{jk} \overset{\mathcal{T}_2}{\to} X_0 \), by Theorem 3.7 and Definition 3.12, we have \( X_0 \in \mathcal{L}_{\mathcal{T}_2}^E \).

By Theorem 3.13, we have \( \mathcal{L}_{\mathcal{T}_2}^E(X_{jk}) = \mathcal{C}_{\mathcal{T}_2}^E(X_{jk}) = \{X_0\} \).

**Theorem 3.15.** Let \( \mathcal{T}_2 \) be an admissible ideal on \( \mathbb{N} \times \mathbb{N} \). Then the set \( \mathcal{C}_{\mathcal{T}_2}^E \) is closed in \( (E, \|\cdot\|) \), for every double sequence \( \{X_{jk}\} \) in \( E \).

**Proof.** Let \( W \in \overline{\mathcal{C}_{\mathcal{T}_2}^E}(X_{jk}) \). Let \( \epsilon > 0 \) and a non-zero \( Z \in E \) be given. Then there exists an \( X_0 \in \mathcal{C}_{\mathcal{T}_2}^E(X_{jk}) \cap U_W(\epsilon, 0) \). Choose \( \eta > 0 \) such that \( U_{X_0}(\eta, 0) \subset U_W(\epsilon, 0) \). Obviously we have

\[
\{(j, k) \in \mathbb{N} \times \mathbb{N} : ||X_{jk} - X_0, Z||_0^+ < \eta \} \subset \{(j, k) \in \mathbb{N} \times \mathbb{N} : ||X_{jk} - W, Z||_0^+ < \epsilon \}.
\]

This implies that \( \{(j, k) \in \mathbb{N} \times \mathbb{N} : ||X_{jk} - W, Z||_0^+ < \epsilon \} \notin \mathcal{T}_2 \). Thus \( W \in \mathcal{C}_{\mathcal{T}_2}^E(\{X_{jk}\}) \). Hence \( \mathcal{C}_{\mathcal{T}_2}^E(X_{jk}) \) is closed in \( E \). \( \Box \)

### 4. \( \mathcal{T}_2^E \) - and \( \mathcal{T}_2^E \)-double Cauchy sequences in fuzzy 2-normed spaces

In this section we study the concepts of \( \mathcal{T}_2 \)-Cauchy and \( \mathcal{T}_2 \)-Cauchy double sequences in \( (E, \|\cdot\|) \). Moreover, we will study the relations between them. The investigation of ideal Cauchy sequences (and nets) was done in [10, 13, 34].

**Definition 4.1.** Let \( (E, \|\cdot\|) \) be a fuzzy 2-normed space and \( \mathcal{T}_2 \) be an admissible ideal of \( \mathbb{N} \times \mathbb{N} \).

A double sequence \( \{X_{jk}\} \) of elements in \( E \) is said to be

1. an \( \mathcal{T}_2^E \)-Cauchy sequence in \( E \) if for every \( \epsilon > 0 \) and a nonzero \( Z \in E \), there exist \( s = s(\epsilon), t = t(\epsilon) \) such that

\[
\{(j, k) \in \mathbb{N} \times \mathbb{N} : ||X_{jk} - X_{st}, Z||_0^+ \geq \epsilon \} \in \mathcal{T}_2.
\]

2. an \( \mathcal{T}_2^E \)-Cauchy sequence in \( E \) if for every \( \epsilon > 0 \) and a nonzero \( Z \in E \), there exists

\[
K = \{(j_m, k_m) : j_1 < j_2 < \cdots : k_1 < k_2 < \cdots \} \subset \mathbb{N} \times \mathbb{N}
\]

such that \( K \in \mathcal{F}(\mathcal{T}_2) \) and \( \{X_{j_mk_m}\} \) is an ordinary \( E \)-Cauchy sequence in \( E \).

The next theorem gives a relation between \( \mathcal{T}_2^E \)- and \( \mathcal{T}_2^E \)-double Cauchy sequences.

**Theorem 4.2.** Let \( (E, \|\cdot\|) \) be a fuzzy 2-normed space and \( \mathcal{T}_2 \) be an admissible ideal of \( \mathbb{N} \times \mathbb{N} \). If \( \{X_{jk}\} \) is an \( \mathcal{T}_2^E \)-double Cauchy sequence, then \( \{X_{jk}\} \) is an \( \mathcal{T}_2^E \)-double Cauchy sequence.

**Proof.** Since \( \{X_{jk}\} \) be an \( \mathcal{T}_2^E \)-double Cauchy sequence, for any \( \epsilon > 0 \) and any non-zero \( Z \in E \), there exist

\[
K = \{(j_m, k_m) : j_1 < j_2 < \cdots : k_1 < k_2 < \cdots \} \in \mathcal{F}(\mathcal{T}_2)
\]

and a number \( n_0 \in \mathbb{N} \) such that

\[
||X_{j_mk_m} - X_{j_mk}, Z||_0^+ < \epsilon
\]

for every \( m, p \geq n_0 \). Now, fix \( p = j_{n_0+1}, r = k_{n_0+1} \). Then for every \( \epsilon > 0 \) and a non-zero \( Z \in E \), we have

\[
||X_{j_mk_m} - X_{pr}, Z||_0^+ < \epsilon
\]
for every $m \geq n_0$. Let $H = N \times N \setminus K$. It is obvious that $H \in \mathcal{F}(I_2)$ and

$$A(\epsilon) = \{(j,k) \in N \times N : \|X_{jk} - X_{pr},Z\|_0^+ \geq \epsilon\} \subset H \cup \{j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots\} \in I_2.$$

Therefore, for every $\epsilon > 0$ and non-zero $Z \in E$, we can find $(p,r) \in N \times N$ such that $A(\epsilon) \in I_2$, i.e., \{X_{jk}\} is an $I_2^E$-double Cauchy sequence.

**Theorem 4.3.** Let $(E, \|\|,\|\|)$ be a fuzzy 2-normed space and $I_2$ be an admissible ideal of $N \times N$. If a sequence \{X_{jk}\} is $I_2^E$-convergent, then it is an $I_2^E$-double Cauchy sequence.

**Proof.** Suppose that $X_{jk} \xrightarrow{I_2^E} X_0$. Then for each $\epsilon > 0$ and a non-zero $Z \in E$, we have

$$A(\epsilon) = \{(j,k) \in N \times N : \|X_{jk} - X_0, Z\|_0^+ \geq \epsilon\} \subset H \cup \{j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots\} \in I_2.$$

Since $I_2$ is an admissible ideal, there exists an $(j_0,k_0) \in N \times N$ such that $(j_0,k_0) \notin A(\epsilon)$. Let $A_1(\epsilon) = \{(j,k) \in N \times N : \|X_{jk} - X_{j_0k_0}, Z\|_0^+ \geq 2\epsilon\}.$

Since $\|\|,\|\|_0^+$ is a 2-norm in the usual sense, we get

$$\|X_{jk} - X_{j_0k_0}, Z\|_0^+ \leq \|X_{jk} - X_0, Z\|_0^+ + \|X_{j_0k_0} - X_0, Z\|_0^+. $$

Observe that if $(j,k) \in A_1(\epsilon)$, then

$$\|X_{jk} - X_0, Z\|_0^+ + \|X_{j_0k_0} - X_0, Z\|_0^+ \geq 2\epsilon.$$

On the other hand, since $(j_0,k_0) \notin A(\epsilon)$, we have

$$\|X_{j_0k_0} - X_0, Z\|_0^+ < \epsilon.$$

So we can conclude that \[\|X_{jk} - X_0, Z\|_0^+ \geq \epsilon, \text{ hence } (j,k) \in A(\epsilon).\] This implies that $A_1(\epsilon) \subset A(\epsilon)$, for each $\epsilon > 0$ and a non-zero $Z \in E$. This gives $A_1(\epsilon) \in I_2$ which shows that \{X_{jk}\} is an $I_2^E$-double Cauchy sequence. \hfill \Box

**Theorem 4.4.** Let $(E, \|\|,\|\|)$ be a fuzzy 2-normed space, $I_2$ an admissible ideal in $N \times N$, \{X_{jk}\} a double sequence in $E$, and $A_{nm} = \{(j,k) \in N \times N : \|X_{jk} - X_{nm}, Z\|_0^+ \geq \epsilon\}$, where $(n,m) \in N \times N$. If \{X_{jk}\} is an $I_2^N$-double Cauchy sequence, then for every $\epsilon > 0$, there exists $B \subset N \times N$ with $B \in I_2$ such that $\|X_{jk} - X_{lt}, Z\|_0^+ < \epsilon$, for all $(j,k), (l,t) \notin B$.

**Proof.** Let $\epsilon > 0$ and a non-zero $Z \in E$ be given. Set $B = A_{nm}(\frac{\epsilon}{2})$, where $(n,m) \in N \times N$. Since \{X_{jk}\} is a double sequence in $E$, we have $B \in I_2$ and for all $(j,k), (l,t) \notin B$, we get

$$\|X_{jk} - X_{nm}, Z\|_0^+ < \frac{\epsilon}{2} \text{ and } \|X_{nm} - X_{lt}, Z\|_0^+ < \frac{\epsilon}{2}.$$

Then we have $\|X_{jk} - X_{lt}, Z\|_0^+ < \epsilon$, for all $(j,k), (l,t) \notin B$, by the triangle inequality, because $\|\|,\|\|$ is a 2-norm in the usual norm. \hfill \Box

Now we will prove that $I_2^E$-convergent implies $I_2^E$-Cauchy condition in a fuzzy 2-normed space.

**Theorem 4.5.** Let $(E, \|\|,\|\|)$ be a fuzzy 2-normed space and $I_2$ be an admissible ideal of $N \times N$. If a double sequence \{X_{jk}\} is $I_2^E$-convergent, then it is an $I_2^E$-double Cauchy sequence.
Proof. By assumption there exists a set

$$K = \{(j_m, k_m) : j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots \} \subset \mathbb{N} \times \mathbb{N}$$

such that $K \in \mathcal{F}(\mathcal{I}_2)$ and $E - \lim_{m} \|X_{j_m k_m} - X_0, Z\|^+_0 = 0$ for each nonzero $Z \in E$, i.e., there exists $n_0 \in \mathbb{N}$ such that $\|X_{j_m k_m} - X_0, Z\|^+_0 < \varepsilon$ for every $\varepsilon > 0$, each non-zero $Z \in E$ and $m > n_0$. Since

$$\|X_{j_m k_m} - X_{j_p k_p}, Z\|^+_0 \leq \|X_{j_m k_m} - X_0, Z\|^+_0 + \|X_{j_p k_p} - X_0, Z\|^+_0$$

for every $\varepsilon > 0$, each non-zero $Z \in E$ and $m, p > n_0$, we have

$$\|X_{j_m k_m} - X_{j_p k_p}, Z\|^+_0 \geq \varepsilon$$

i.e., $\{X_{jk}\}$ is an $I_2^E$-double Cauchy sequence in $E$. Then by Theorem 4.2, $\{X_{jk}\}$ is an $I_2^E$-double Cauchy sequence.

**Theorem 4.6.** Let $(E, ||\cdot\||)$ be a fuzzy 2-normed space and $\mathcal{I}_2$ be an admissible ideal in $\mathbb{N} \times \mathbb{N}$ with property (AP). Then the concepts $I_2^E$-double Cauchy sequence and $I_2^E$-double Cauchy sequence coincide.

**Proof.** If $\{X_{jk}\}$ is an $I_2^E$-double Cauchy sequence, then it is an $I_2^E$-double Cauchy sequence by Theorem 4.2 (even if $\mathcal{I}_2$ does not have the (AP) property).

So, we have to prove the converse. Let $\{X_{jk}\}$ be an $I_2^E$-double Cauchy sequence. Then by definition, there exist an $j_0 = j_0(\varepsilon), k_0 = k_0(\varepsilon)$ such that

$$A(\varepsilon) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_{j_0 k_0}, Z\|^+_0 \geq \varepsilon\} \subset \mathcal{I}_2$$

for every $\varepsilon > 0$ and non-zero $Z \in E$.

Let $P_i = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|X_{jk} - X_{s_i t_i}, Z\|^+_0 < \frac{1}{i}\} = 1, 2, \cdots$, $s_i = j_0(\frac{1}{i}), t_i = k_0(\frac{1}{i})$. It is clear that $P_i \in \mathcal{F}(\mathcal{I}_2)$ for $i = 1, 2, \cdots$. Since $\mathcal{I}_2$ has the property (AP), then by Lemma 3.6 there exists a set $P \subset \mathbb{N} \times \mathbb{N}$ such that $P \in \mathcal{F}(\mathcal{I}_2)$, and $P \setminus P_i$ is finite for all $i$. Now we prove that

$$\lim_{j, k, s, t \rightarrow \infty} \|X_{jk} - X_{st}, Z\|^+_0 = 0.$$
5. Conclusion

In recent years the study of fuzzy numbers and fuzzy normed spaces related to convergence has attracted a big number of works and a wide variety of approaches was developed. In this paper we have focused on convergence of double sequences in fuzzy 2-normed spaces with respect to an ideal on \( \mathbb{N} \times \mathbb{N} \) and proved several results. We think that it may be interesting to make a similar investigation for convergence of double sequences in 2-fuzzy 2-normed spaces and related structures, as well as in (non-Archimedean) fuzzy anti-2-normed spaces.

6. Acknowledgment

We are grateful to the referees for useful comments and suggestions.

References


Ljubiša D.R. Kočinac is a Professor Emeritus in the Department of Mathematics of the Faculty of Sciences and Mathematics at the University of Niš, Serbia since 2014. His current research interests include General Topology, Topological Groups and Real Analysis.

Mohammad H.M. Rashid is an Associate Professor in the Department of Mathematics and Statistics of the Faculty of Science at the University of Mut‘ah, Jordan. Currently he is Head of the Department of Mathematics and Statistics. The major field of his research interests is Functional Analysis/Operator Theory.