TWMS J. Pure Appl. Math., V.9, N.2, 2018, pp.173-189

ON C_u^*- AND C_u- EMBEDDED UNIFORM SPACES

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Abstract. For a uniform space uX the concept of C_u-embedding (C_u^*-embedding) in some uniform space is introduced. An analogue of Urysohn's Theorem is proved and it is established, that uX is C_u^*-embedded in the Wallman $\beta-$like compactification $\beta_u X$, and any compactification of uX in which uX is C_u^*-embedded, must be $\beta_u X$. A uniformly realcompact space is determined. It is proved, that uX is C_u-embedded in the Wallman realcompactification $\upsilon_u X$, and any uniform realcompactification in which uX is C_u-embedded, is $\upsilon_u X$.

Key words: $u-$open (closed) sets, z_u-filter, z_u-ultrafilter, $coz-$mapping, $u-$continuous function, Wallman $\beta-$like compactification, Wallman realcompactification.

AMS Subject Classification: 54C20, 54C45, 54D30, 54D60, 54E15, 54 H13 .

1. Introduction

Extensions of (bounded) continuous and (bounded) uniformly continuous functions from subspaces of topological and uniform spaces to the whole space are the most important and actual problems ([2, 8]). For topological spaces the concepts of C^*-embeddings and C-embeddings of their subspaces, introduced by C. Kohls [13, Notes, Chapter 1] and P. Urysohn, allowed one to prove that to be a normal space is equivalent to that every closed subspace is $C(C^*)-$embedded in it [13, Notes, Chapter 1], [10, 2.1.8]. M. Stone and E. Čech proved that a completely regular space X is C^*-embedded in its Stone-Čech compactification βX, and any compactification of X in which X is C^*-embedded must be βX [13, Th. 6.5 (II)]. E. Hewitt proved that a completely regular space X is $C-$embedded in its realcompactification υX, and any realcompactification of X in which X is $C-$embedded must be υX [13, Th. 8.7 (II)]. M. Katetov [18] proved that any bounded uniformly continuous function on a uniform subspace can be extended on the whole space.

In [8], for a uniform space uX the Wallman $\beta-$like compactification $\beta_u X$ and the Wallman realcompactification $\upsilon_u X$ have been constructed and their uniformities described. Since $U(uX) \subset C_u(X) \subset C(X)$ and $U^*(uX) \subset C_u^*(uX) \subset C^*(X)$, the concepts of C_u-embedding and (C_u^*-)embedding of a uniform subspaces are naturally determined (Definition 3.2) and analogues of Urysohn's Theorem (Theorem 3.4) and Theorem on C_u-embedding of a C_u^*-embedded subspaces are proved (Theorem 3.5). For the Wallman $\beta-$like compactification $\beta_u X$ of a uniform space uX an analogue of Stone-Čech Theorem is proved: a uniform space uX is C_u^*-embedded in the Wallman $\beta-$like compactification $\beta_u X$ and any compactification of uX, in which uX is C_u^*-embedded must be $\beta_u X$ (Theorem 3.7, Corollary 3.5, Theorem 3.8). It is also proved that a unform subspace $u'S$ of the uniform space uX is C_u^*-embedded in uX if and only if

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Manuscript received Mart 2016.

$[S]_{\beta_u X} = \beta_{u'} S$ (Proposition 3.3). An example of a uniform space uX which is C^*_u-embedded, but is not C^*-embedded in $\beta_u X$ is given (Remark 3.2).

The concept of uniformly realcompact uniform space is introduced (Definition 4.2), of its some properties are studied and the example of a uniform space uX which is C_u-embedded, but is not $C-$embedded in the Wallman realcompactification $v_u X$ is provided (Theorem 4.1, Corollary 4.1, Theorem 4.2, Corollary 4.2, Propositions 4.1 and 4.2, Corollaries 4.2 and 4.3). For a realcompactification $v_u X$ of a uniform space uX an analogue of Hewitt Theorem is proved: uX is C_u-embedded in its Wallman realcompactification $v_u X$, and any uniform realcompactification in which uX is C_u-embedded is $v_u X$ (Theorem 4.4, Corollary 4.4, Theorem 4.5).

2. Notation and preliminaries

All notations and properties of uniform spaces are taken from books [17, 3, 10], a normal bases properties from [11] and constructions using them from books [1, 15, 20], properties of the Stone-Čech compatification and its interrelation with rings of functions from books [13, 23].

For a uniform space uX, where u is the uniformity in terms of uniform coverings, we denote by $U(uX)$ $(U^*(uX))$ the set of all (bounded) uniformly continuous functions on uX. $\mathcal{Z}_u = \{f^{-1}(0) : f \in U(uX)\}$ and, evidently, $\mathcal{Z}_u = \{g^{-1}(0) : g \in U^*(uX)\}$. We note that $U^*(uX)$ is a commutative ring with unit, whereas $U(uX)$, in general, is not so. All sets of $\mathcal{Z}_u$ are said to be *$u-$closed* [5, 6] and all sets of $C\mathcal{Z}_u = \{X \backslash Z : Z \in \mathcal{Z}_u\}$ are said to be *$u-$open* [3]. When $u = u_f$ is the *fine* uniformity on a Tychonoff space X, then $U(u_f X) = C(X)(U^*(u_f X) = C^*(X))$ is the set of all (bounded) continuous functions [13, 10]. $\mathcal{Z}_{u_f} = \mathcal{Z}(X)$ is a family of all zero-sets, and $C\mathcal{Z}_{u_f} = C\mathcal{Z}(X)$ is a family of all cozero-sets [13]. A family (covering) α consisting of $u-$open sets (cozero-sets) is said to be an *$u-$open* (a *cozero*) covering.

$\mathcal{Z}_u$ is the base of closed sets topology, forms *separating, nest-generated intersection ring* on X [21], and hence it is a *normal base* [11, 15].

Definition 2.1. A mapping $f : uX \to vY$ between uniform spaces is said to be a *coz$-$mapping*, if $f^{-1}(C\mathcal{Z}_v) \subseteq C\mathcal{Z}_u$ (or $f^{-1}(\mathcal{Z}_v) \subseteq \mathcal{Z}_u$) [5, 6, 12]. A mapping $f : uX \to Y$ from a uniform space uX into a Tychonoff space Y is said to be *z_u-continuous*, if $f^{-1}(C\mathcal{Z}(X)) \subseteq C\mathcal{Z}_u$ (or $f^{-1}(\mathcal{Z}(Y)) \subseteq \mathcal{Z}_u$) [9].

Evidently, every uniformly continuous mapping is a *coz*$-$mapping, while the converse, generally speaking, is not true [5, 6, 7]. Also, every z_u-continuous mapping $f : uX \to Y$ is a *coz*$-$mapping of $f : uX \to vY$ for any uniformity v on Y. If Y is a Lindelöf space or (Y, ρ) is a metric space, then its *coz*$-$mapping is a z_u-continuous (for example, [5, 6]). If $Y = \mathbb{R}$ or $Y = I$, then the *coz*$-$mapping $f : uX \to \mathbb{R}$ is said to be a *$u-$continuous function* and the *coz*$-$mapping $f : uX \to I$ is said to be a *$u-$function* [5, 6].

We denote by $C_u(X)$ $(C^*_u(X))$ the set of all (bounded) $u-$continuous functions on the uniform space uX and by $\mathcal{Z}(uX)$ the ring of zero-sets functions from $C_u(X)$ or $C^*_u(X)$ and $C\mathcal{Z}(uX)$ consists of complements of sets of $\mathcal{Z}(uX)$ and, vice versa.

The topology of a uniform space is generated by its uniformity, and in case of a compactum X we always use its unique uniformity. The restriction of a uniformity from a uniform space vY to its subspace X is denoted $v|_X$ and the restriction of a function f from Y into $\mathbb{R}$ to a subset $X \subset Y$ is denoted $f|_X$. A uniform space uX which has a base of uniform coverings of cardinality $\leq \tau$ is said to be *$\tau-$bounded* [3, 4].

We denote the set of all natural numbers by $\mathbb{N}$, $\mathbb{R}$ is the real line, uniformity $u_{\mathbb{R}}$ on $\mathbb{R}$ is induced by the ordinary metric; $I = [0, 1]$ is a unit interval; for $X \subset Y$ and a family of subsets

$\mathfrak{F}$ in Y we denote $X \wedge \mathfrak{F} = \{X \cap F : F \in \mathfrak{F}\}$ and by $[X]_Y$ the closure of X in Y. For families of subsets $\mathfrak{F}$ and $\mathfrak{F}'$ we denote $\mathfrak{F} \wedge \mathfrak{F}' = \{F \cap F' : F \in \mathfrak{F}, F' \in \mathfrak{F}'\}$.

A filter $\mathcal{F}$ is called *countably centered* if $\cap_{n\in\mathbb{N}} F_n \neq \emptyset$ for any countable subfamily $\{F_n\}_{n\in\mathbb{N}}$ of the filter $\mathcal{F}$.

The natural uniformity on uX, generated by $U(uX)\,(U^*(uX))$, the $u_c(u_p)$ is the smallest uniformity on X with respect to all its functions from $U(uX)$ $(U^*(uX))$ which are uniformly continuous. Evidently, $u_p \subseteq u_c \subseteq u_\omega \subseteq u$, where a base of uniformity u_ω is formed by all countable uniform coverings of u. The *Samuel compactification* s_uX is a completion of X with respect to the uniformity u_p.

Proposition 2.1. [8] The set $\mathcal{B}_p^*$ $(\mathcal{B}_\omega^*)$ of all finite countable $u-$open coverings of a uniform space uX is the base of uniformity $u_p^z(u_\omega^z)$. Moreover $u_p \subseteq u_p^z$, $u_p \subseteq u_c \subseteq u_\omega \subseteq u_\omega^z$.

Proposition 2.2. [8] $C_u(X)$ forms a complete subring of $C(X)$ with the inversion. It contains constant functions, separates points and closed sets, is uniformly closed and is closed under inversion, i.e. if $f \in C_u(X)$ and $f(x) \neq 0$ for all $x \in X$ then $1/f \in C_u(X)$ (an algebra in sense of [14, 16]).

Lemma 2.1. [8]

(1) *coz*$-$mapping $f : uX \to vY$ into a compact space vY is a uniformly continuous mapping $f : u_p^zX \to vY$;

(1') *coz*$-$mapping $f : uX \to vY$ into $\aleph_0-$bounded uniform space vY is a uniformly continuous mapping $f : u_\omega^zX \to vY$;

(2) $U(uX) = U(u_cX) = U(u_\omega X) \subset U(u_\omega^zX) = C_u(X)$;

(2') $U(u_pX) = U^*(uX) \subset U(u_p^zX) = U^*(u_\omega^zX) = C_u^*(X) \subset C_u(X)$;

(3) $\mathcal{Z}_u = \mathcal{Z}_{u_p} = \mathcal{Z}_{u_c} = \mathcal{Z}_{u_\omega} = \mathcal{Z}_{u_p^z} = \mathcal{Z}_{u_\omega^z} = \mathcal{Z}(uX)$.

(4) $C_u(X)$ is a complete ring of functions with inversion on X.

Let $\omega(X, \mathcal{Z}_u)$ be the Wallman compactification of X with respect to the normal base $\mathcal{Z}_u$ [11, 1, 15].

Theorem 2.1. [8] For a uniform space uX the following compactifications of X coincide:

(1) The completion of X with respect to u_p^z.

(2) The Wallman compactification $\omega(X, \mathcal{Z}_u)$ of X with respect to the normal base $\mathcal{Z}_u$.

(3) The compactification which is the set of all maximal ideals of $C_u^*(X)$ equipped with the Stone topology [22].

We note that $\omega(X, \mathcal{Z}_u)$ is a $\beta-$like compactification of X [19], and we put $\beta_uX = \omega(X, \mathcal{Z}_u)$

Corollary 2.1. [8]

I. Every *coz*$-$mapping $f : uX \to vY$ can be extended to the continuous mapping $\beta_uf : \beta_uX \to \beta_vY$ [1, 15].

II. The first axiom of countability does not hold in any point $x \in \beta_uX \setminus X$ [21].

III. For uniform spaces uX and $u'X$ we have $\beta_uX = \beta_{u'}X$ if and only if $\mathcal{Z}_u = \mathcal{Z}_{u'}$ [21].

Theorem 2.2. [8] For a uniform space uX the following conditions are equivalent:

(1) The Samuel compactification s_uX of uX is a $\beta-$like compactification of X;

(2) $u_p = u_p^z$;

(3) any *coz*$-$mapping $f : uX \to K$ into a compactum K can be extended to s_uX;

(4) any $u-$function $f : uX \to I$ into I can be extended to s_uX;

(5) if $Z_1, Z_2 \in \mathcal{Z}_u$ and $Z_1 \cap Z_2 = \emptyset$, then $[Z_1]_{s_uX} \cap [Z_2]_{s_uX} = \emptyset$;

(6) $[Z_1]_{s_uX} \cap [Z_2]_{s_uX} = [Z_1 \cap Z_2]_{s_uX}$ is fulfilled for any $Z_1, Z_2 \in \mathcal{Z}_u$;

(7) every point of s_uX is the limit point for a unique z_u-ultrafilter on uX;

(8) every z_u-ultrafilter is a Cauchy filter with respect to u_p.

Definition 2.2 [21]. The *Wallman realcompactification* of a uniform space uX is the subspace $\upsilon(X, \mathcal{Z}_u) = \upsilon_u X \subset \beta_u X$ consisting of the set of all countably centered z_u-ultrafilteres on $\mathcal{Z}_u$.

Theorem 2.3. [8] For a uniform space uX the following realcompactifications of X coincide:

(1) the completion of X with respect to u^z_ω;
(2) the Wallman realcompactification $\upsilon_u X = \upsilon(X, \mathcal{Z}_u)$;
(3) the intersection of all cozero-sets in $\beta_u X$ which contain X;
(3') the intersection of all cozero-sets in $s_u X$ which contain X;
(4) the $Q-$closure of X in $\beta_u X$;
(4') the $Q-$closure of X in $s_u X$;
(5) the weak completion $\mu_{u^z_\omega} X$ coincides with the completion of X with respect to u^z_ω;
(6) the weak completion $\mu_{u_\omega} X$ of X with respect to u_ω;
(7) the weak completion $\mu_{u_c} X$ of X with respect to u_c.

Let X^υ be some realcompactification of X, u^υ_ω be a uniformity on X^υ whose base consists of all countable cozero-sets coverings, u^υ_c be the smallest uniformity on X^υ in which all functions from $C(X^\upsilon)$ are uniformly continuous [10, 8.19, 8.1.D, 8.1.I, 8.3.19, 8.3.F], $\mathcal{Z}(C(X^\upsilon))$ be the ring of zero-sets of functions from $C(X^\upsilon)$ and $\mathcal{Z}_{X^\upsilon} = X \wedge \mathcal{Z}(C(X^\upsilon))$.

Theorem 2.4. [8] For a realcompactification X^υ of X the following conditions are equivalent:

(1) X^υ is the completion of X with respect to $u_\omega = u^\upsilon_\omega|_X$;
(2) X^υ is the weak completion $\mu_{u_c} X$of X with respect to $u_c = u^\upsilon_c|_X$;
(3) X^υ is the Wallman realcompactification $\upsilon_{u_\omega}(X, \mathcal{Z}_{X^\upsilon})$ of X with respect to $\mathcal{Z}_{X^\upsilon}$;
(4) any $z_{u_\omega}-$continuous mapping $f : u_\omega X \to Y$ into a realcompact space Y has $z_{u^\upsilon_\omega}-$continuous extension to X^υ;
(5) any $z_{u_\omega}-$continuous function $f : u_\omega X \to \mathbb{R}$ has $z_{u^\upsilon_\omega}-$continuous extension to X^υ;
(6) for any $\{Z_i\}_{i\in\mathbb{N}} \subset \mathcal{Z}_{X_\upsilon}$ such that $\cap_{i\in\mathbb{N}} Z_i = \emptyset$ it follows $\cap_{i\in\mathbb{N}}[Z_i]_{X^\upsilon} = \emptyset$;
(7) $\cap_{i\in\mathbb{N}}[Z_i]_{X^\upsilon} = [\cap_{i\in\mathbb{N}} Z_i]_{X^\upsilon}$ is fulfilled for any $\{Z_i\}_{i\in\mathbb{N}} \subset \mathcal{Z}_{X_\upsilon}$;
(8) each point in X^υ is the limit of a unique countably centered $z_{u_\omega}-$ultrafilter on X.

For the interrelations of $u-$closed set filters with the ideals of rings $C_u(X)(C^*_u(X))$ by analogy with Chapter 2 of [13] we introduce the next notions.

Definition 2.3. A nonempty subfamily $\mathcal{F}$ of $\mathcal{Z}_u$ is said to be a z_u-*filter* on uX provided that (i) $\emptyset \notin \mathcal{F}$; (ii) if $Z_1, Z_2 \in \mathcal{F}$, then $Z_1 \cap Z_2 \in \mathcal{F}$; (iii) if $Z \in \mathcal{F}$, $Z' \in \mathcal{Z}_u$ and $Z \subset Z'$, then $Z' \in \mathcal{F}$.

A natural mapping $\mathbf{Z} : C_u(X) \to \mathcal{Z}_u$, where for any $f \in C_u(X)$, $\mathbf{Z}(f) = \{x \in X : f(x) = 0\} = f^{-1}(0) \in \mathcal{Z}_u$, is determined.

Theorem 2.5. If I is an ideal of the ring $C_u(X)$, then the family $\mathbf{Z}(I) = \{\mathbf{Z}(f) : f \in I\}$ is a z_u-filter on uX, and, vice versa, if $\mathcal{F}$ is a z_u-filter on uX, then the family $\mathbf{Z}^{-1}(\mathcal{F}) = \{f : \mathbf{Z}(f) \in \mathcal{F}\}$ is an ideal in $C_u(X)$.

Proof. It is similar to [13, Th. 2.3]. □

By a z_u-*ultrafilter* on uX is meant a maximal z_u-filter, i.e. one not contained in any other z_u-filters.

Theorem 2.6. If I is a maximal ideal of the ring $C_u(X)$, then $\mathcal{Z}(I)$ is a z_u-ultrafilter on uX and if p is a z_u-ultrafilter on uX, then $\mathbf{Z}^{-1}(p)$ is a maximal ideal in $C_u(X)$. The mapping $\mathbf{Z} : C_u(X) \to \mathcal{Z}_u$ is one-to-one from the set of all maximal ideals in $C_u(X)$ onto the set of all z_u-ultrafilters

Proof. It is analogically to [13, Th. 2.5]. □

Theorem 2.7.

(a) Let I be a maximal ideal in the ring $C_u(X)$. If $\mathbf{Z}(f)$ meets every member of $\mathbf{Z}(I)$, then $f \in I$.
(b) Let p be a z_u-ultrafilter on uX. If an $u-$closed set Z meets every member of p, then $Z \in p$.

Proof. The proof is similar to that of [13, Th. 2.6]. □

Theorem 2.8. Let I be an ideal in $C_u(X)$ such that if $\mathbf{Z}(f) \in \mathbf{Z}(I)$, then it implies $f \in I$. Then the next statements are equivalent:

(1) I is prime.
(2) I contains a prime ideal.
(3) For all $g, h \in C_u(X)$, if $gh = 0$, then $g \in I$ or $h \in I$.
(4) For every $f \in C_u(X)$ there is an $u-$closed set $\mathbf{Z}(f)$ on which f does not change sign.

Proof. Analogically to [13, Th. 2.9]. □

Theorem 2.9. Every prime ideal in $C_u(X)$ is contained in a unique maximal ideal.

Proof. Similarly to [13, Th. 2.11]. □

Definition 2.4. Let $\mathcal{F}$ be a z_u-filter. If $Z_1, Z_2 \in \mathcal{Z}_u$ and from $Z_1 \cup Z_2 \in \mathcal{F}$ it follows that either $Z_1 \in \mathcal{F}$ or $Z_2 \in \mathcal{F}$, then $\mathcal{F}$ is said to be a *prime* z_u-*filter*.

Theorem 2.9.

(a) If I is a prime ideal in $C_u(X)$, then $\mathbf{Z}(I)$ is a prime z_u-filter.
(b) If $\mathcal{F}$ is a prime z_u-filter, then $\mathbf{Z}^{-1}(\mathcal{F})$ is a prime z_u-ideal.

Proof. It is analogically to [13, Th. 2.12]. □

Corollary 2.2. Every prime z_u- filter is contained in a unique z_u-ultrafilter.

Proof. It immediately follows from the Theorems 2.6 and 2.9. □

Definition 2.4. Let I be any ideal in $C_u(X)$ (or $C_u^*(X)$). If $\cap\mathcal{Z}(I) \neq \emptyset$, then I is said to be a *fixed* ideal; otherwise, I is said to be a *free* ideal.

Theorem 2.10. The following statements are equivalent:

(1) uX is a compact uniform space.
(2) Every ideal in $C_u(X)$ is fixed, i.e. every z_u-filter is fixed
(2') Every ideal in $C_u^*(X)$ is fixed.
(3) Every maximal ideal in $C_u(X)$ is fixed, i.e. every z_u-ultrafilter is fixed
(3') Every maximal ideal in $C_u^*(X)$ is fixed.

Proof. It is analogically to [13, Th. 4.11]. □

Lemma 2.2. Let $f : uX \to vY$ be a *coz*$-$ mapping and let $\mathcal{F}$ be a prime z_u-filter on uX. Then $f^\sharp(\mathcal{F}) = \{Z \in \mathcal{Z}_v : f^{-1}(Z) \in \mathcal{F}\}$ is a prime z_v-filter on vY.

Proof. It is analogically to [13, Th. 4.12]. □

3. C_u^*- EMBEDDING IN $\beta-$ LIKE COMPACTIFICATIONS

Definition 3.1. Two subsets A and B of uX are said to be $u-$*separated* in uX if there exists a $u-$function $f : uX \to I$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.

Remark 3.1. If $Z_1, Z_2 \in \mathcal{Z}_u$ on uX and $Z_1 \cap Z_2 = \emptyset$, then the function $f(x) = g_1(x)/(g_1(x)+g_2(x))$ is a $u-$function [5, 6], where $g_i : uX \to I$ are uniformly continuous functions, $Z_i = g_i^{-1}(0)$, $(i = 1, 2)$ and $f(Z_1) = \{0\}, f(Z_2) = \{1\}$. Any segment $[-r, r]$ is uniformly homeomorphic to I. Let $h : I \to [-r, r]$ be a uniform homeomorphism such that $h(0) = \{-r\}, h(1) = \{r\}$. Then the function $F : uX \to [-r, r]$, where $F = h \circ f$, is $u-$continuous and $F(Z_1) = \{-r\}, F(Z_2) = \{r\}$.

Theorem 3.1. Two sets in uX are $u-$ separated if and only if they are contained in disjoint $u-$ closed sets. Moreover, $u-$separated sets have disjoint $u-$closed neighborhoods.

Proof. Let A and B be $u-$separated in uX. Then there exists $u-$function $f : uX \to I$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$. The sets $Z_1 = \{x : f(x) \leq 1/3\}$ and $Z_2 = \{x : f(x) \geq 2/3\}$ are $u-$closed neighborhoods of A and B, respectively, and $Z_1 \cap Z_2 = \emptyset$.

Conversely, if $A \subset Z_1$, $B \subset Z_2$, $Z_i \in \mathcal{Z}_u, i = 1, 2$, and $Z_1 \cap Z_2 = \emptyset$, then, according to Remark 3.2, there exists a $u-$ function $f : uX \to I$ such that $f(x) = 0$ for all $x \in Z_1$ and $f(x) = 1$ for all $x \in Z_2$. Hence A and B are $u-$separated in uX. □

Corollary 3.1. If A and B are $u-$ separated in uX, then there exist $u-$closed sets F and Z such that $A \subset X \setminus Z \subset F \subset X \setminus B$.

Proof. Let $f : uX \to I$ be a $u-$function such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$. Put $F = \{x : f(x) \leq 1/3\}$ and $Z = \{x : f(x) \geq 1/3\}$. Then F and Z are $u-$closed sets and it is easy to check, that the condition of this corollary is fulfilled. □

Corollary 3.2. Every neighborhood of a point in a uniform space uX contains a $u-$closed-neighborhood of the point.

Proof. Let $x \in X$ be an arbitrary point and $x \in O$ be an arbitrary open neighborhood of the point x. Then $x \notin F = X \setminus O$ and F is a closed set in X. Hence, there exists a uniformly continuous function $f : uX \to I$ such that $f(x) = 0$ and $f(x) = 1$ [3, 10, 17]. Every uniformly continuous function is $u-$continuous, hence, x and F are $u-$separated and the remaining follows from Corollary 3.4. □

Let uX be a uniform space. A point $x \in X$ is said to be a *cluster point* of a z_u- filter $\mathcal{F}$ if every neighborhood of x meets every member of $\mathcal{F}$. The z_u-filter $\mathcal{F}$ is said to *converge* to the *limit* x if every neighborhood of x contains a member of $\mathcal{F}$.

Proposition 3.1. A z_u-filter $\mathcal{F}$ converges to x if and only if $\mathcal{F}$ contains the z_u-filter of all $u-$closed-neighborhoods of x. If x is a cluster point of a z_u-filter $\mathcal{F}$, then at least one z_u-ultrafilter containing $\mathcal{F}$ converges to x.

Proof. It is analogically to [13, Th.3.16]. □

Theorem 3.2. Let uX be a uniform space, $x \in X$ and let $\mathcal{F}$ be a prime z_u-filter on uX. The following conditions are equivalent:

(1) x is a cluster point of $\mathcal{F}$.
(2) $\mathcal{F}$ converges to x.
(3) $\cap\mathcal{F} = \{x\}$.

Proof. It is analogically to [13, Th.3.17]. □

Theorem 3.3. Let p_x be a family of all $u-$closed sets containing a given point x of a uniform space uX. Then

(a) x is a cluster point of a z_u-filter $\mathcal{F}$ if and only if $\mathcal{F} \subset p_x$.
(b) p_x is the unique z_u-ultrafilter converging to x.
(c) Distinct z_u-ultrafilters cannot have a common cluster point.
(d) If $\mathcal{F}$ is a z_u-filter converging to x, then p_x is the unique z_u-ultrafilter containing $\mathcal{F}$.

Proof. It immediately follows from 3.5, 3.6, 3.7. □

Definition 3.2. Let X be a subspace of a Tychonoff space Y and u be a uniformity on X, v be a uniformity on Y such that $\mathcal{Z}_v \wedge X = \mathcal{Z}_u$. The uniform space uX is said to be $C_u(C_u^*)-$*embedded* in the uniform space vY, if any function of $C_u(X)$ $(C_u^*(X))$ can be extended to a function in $C_v(Y)$ $(C_v^*(Y))$.

Theorem 3.4. (Urysohn's Extension Theorem) Let X be a subspace of a Tychonoff space Y, u be a uniformity on X and v be a uniformity on Y such that $\mathcal{Z}_v \wedge X = \mathcal{Z}_u$. Then uX is C_u^*-embedded in vY if and only if any two $u-$separated sets in X are $v-$separated in Y.

Proof. Necessity. If A and B are $u-$separated sets in uX, there exists a function f in $C_u^*(X)$ that is equal to 0 on A and 1 on B. By hypothesis, f has an extension to a function g in $C_v^*(Y)$. Since g is 0 on A and 1 on B, these sets are $u-$separated sets in vY.

Sufficiency. Let f_1 be a given function in $C_u^*(X)$. Then $|f_1| \leq m$ for some $m \in \mathbb{N}$. Define $r_n = (m/2)(2/3)^n$, $n \in \mathbb{N}$. Then $|f_1| \leq m = 3r_1$. Inductively, given $f_n \in C_u^*(X)$ with $|f_n| \leq 3r_n$, define $A_n = \{x \in X : f_n(x) \leq -r_n\}$ and $B_n = \{x \in X : f_n(x) \geq r_n\}$. Then A_n and B_n are $u-$closed sets in uX and $A_n \cap B_n = \emptyset$. Then, by Remark 3.1, A_n and B_n are $u-$separated in uX. Accordingly, there exists a function g_n in $C_v^*(Y)$ equal to $-r_n$ on A_n and $2r_n$ on B_n with $|g_n| \leq r_n$. The values of f_n and g_n on A_n lie between $-3r_n$ and $-r_n$; on B_n, they lie between r_n and $3r_n$; and elsewhere on X they are between $-r_n$ and r_n. Let $f_{n+1} = f_n - g_n|_X$ and we have $|f_{n+1}| \leq 2r_n$, i.e. $|f_{n+1}| \leq 3r_{n+1}$. This completes the induction step.

Now put $g(x) = \Sigma_{n\in\mathbb{N}} g_n(x), x \in X$. Since the series $\Sigma_{n\in\mathbb{N}} r_n$ converges uniformly and since $|g_n| \leq r_n$ for all $n \in \mathbb{N}$, it follows that $\Sigma_{n\in\mathbb{N}} g_n(x)$ converges uniformly and, by Proposition 2.3, g is a bounded $v-$continuous function, i.e. $g \in C_v^*(Y)$. For every $n \in \mathbb{N}$ we have $(g_1+g_2+...+g_n)|_X = (f_1 - f_2) + (f_2 - f_3) + ... + (f_n - f_{n+1})$, i.e. $(g_1 + g_2 + ... + g_n)|_X = f_1 - f_{n+1}$. Since the sequence $\{f_{n+1}(x) : n \in \mathbb{N}\}$ approaches 0 at every x of X, this shows that $g(x) = f_1(x)$ for all $x \in X$. Thus, g is a $v-$continuous extension of f_1. □

Corollary 3.2. Let uX be a uniform subspace of vY. Then uX is C_u^*-embedded in vY if and only if any two $u-$separated sets in X are $v-$separated in Y.

Proof. It immediately follows from Theorem 3.5, since $u = v|_X$, hence $\mathcal{Z}_v \wedge X = \mathcal{Z}_u$. □

Corollary 3.3. Let X be a subspace of a Tychonoff space Y, S be a subspace of X, u be a uniformity on X, v a uniformity on Y and w a uniformity on S such that $\mathcal{Z}_w = \mathcal{Z}_u \wedge S$ both $\mathcal{Z}_u = \mathcal{Z}_v \wedge X$ and uX is C_u- (C_u^*-)embedded in vY. Then wS is C_w- $(C_w^*)-$embedded in vY if and only if wS is C_w- $(C_w^*)-$embedded in uX.

Proof. Let wS be $C_w(C_w^*)-$embedded in vY, i.e. any $w-$continuous (bounded) function $f \in C_w(X)(C_w^*(X))$ can be extended to $v-$continuous function $g \in C_v(Y)$ $(C_v^*(Y))$. It is correctly, as $\mathcal{Z}_w = \mathcal{Z}_v \wedge X$. Evidently, $h : g|_X \in C_u(X)$ $(C_u^*(X))$ and h is a $u-$continuous (bounded) extension of the function f. The converse statement is obvious. □

Theorem 2.4. A C_u^*-embedded subset is C_u-embedded if and only if it is $u-$separated from every $u-$closed set disjoint from it.

Proof. Let uX be C_u^*-embedded in vY. Let $\mathbf{Z}(h) = h^{-1}(0)$ be $v-$closed in Y and $\mathbf{Z}(h) \cap X = \emptyset$. Then $h(x) \neq 0$ for all $x \in X$. Then, by Proposition 2.2, the function $f(x) = 1/h(x)$ is $u-$continuous for all $x \in X$, i.e. $f \in C_u(X)$. Let g be a $u-$continuous extension of f on X. Then $g \cdot h$ belongs to $C_v(Y)$ (Proposition 2.2) and equals to 1 on X and 0 on $\mathbf{Z}(h)$.

Conversely, let $f \in C_u(X)$ be an arbitrary function. Then $arctg \circ f : uX \to (-\pi/2; \pi/2)$ is a $u-$continuous bounded function on uX, i.e. $arctg \circ f \in C_u^*(X)$. Let g be a $v-$continuous extension of $arctg \circ f$, i.e. $g \in C_v(Y)$. A set $Z = \{x \in Y : |g(x)| \geq \pi/2\}$ is $v-$closed and $Z \cap X = \emptyset$. By the condition there exists a function $h \in C_v(Y)$ which is equal to 1 on X and 0 on Z, $|h| \leq 1$. A function $g \cdot h$ is $v-$continuous, by Proposition 2.3, and $g \cdot h|_X = arctg \circ f$ and $|(g \cdot h)(x)| < \pi/2$ for all $x \in Y$. Thus, $tg \circ (g \cdot h)$ is a $v-$continuous extension of f on Y. □

Definition 3.3. Let uX be a uniform space, and X dense in a Tychonoff space Y. A point $x \in Y$ is a *cluster point* of z_u-filter $\mathcal{F}$ on uX, if every neighborhood of point x in Y meets every member of $\mathcal{F}$, and x is a *cluster point* of $\mathcal{F}$ provided that $p \in \cap\{[Z]_Y : Z \in \mathcal{F}\}$.
We will say, that z_u-filter $\mathcal{F}$ *converges to a limit* x, if every neighborhood of point x in Y contains a member of $\mathcal{F}$.

Lemma 3.1. Let uX be a uniform space, X be dense in a Tychonoff space Y and v be a uniformity on Y such that $\mathcal{Z}_v \wedge X = \mathcal{Z}_u$. If Z is a $u-$closed set in uX and $x \in [Z]_Y$, then at least one z_u-ultrafilter on uX contains Z and converges to x.

Proof. Let $\mathcal{F}$ be a z_v-filter on vY of all $v-$closed-set-neighborhoods of x and $\mathcal{F}' = \mathcal{F} \wedge X$. Since $x \in [Z]_Y$, $\mathcal{F}' \cup \{Z\} \subseteq Z_u$ has the finite intersection property, and so is contained in a z_u-ultrafilter p_x. Clearly p_x converges to x. □

Corollary 3.3. Under conditions of Lemma 3.1, every point in Y is the limit of at least one z_u-ultrafilter on uX.

Proof. It immediately follows from Lemma 3.1 under $Z = X$. □

Theorem 3.6. Let uX be a uniform space, X be dense in a Tychonoff space Y, and v be a uniformity on Y such that $\mathcal{Z}_v \wedge X = \mathcal{Z}_u$, and every point of Y be a limit of unique z_u-ultrafilter on uX. Then on a Y there exists a precompact uniformity v_p^z such that $v_p^z|_X = u_p^z$.

Proof. For any point $x \in Y$, p_x is a unique z_u-ultrafilter on uX, converging to x. Let $Z \in \mathcal{Z}_u$ be an arbitrary member. Put $\overline{Z} = \{x \in Y : Z \in p_x\}$.

Lemma 3.2. Under conditions of Theorem 3.6, if $Z \in \mathcal{Z}_u$, then the set $\overline{Z} = \{x \in Y : Z \in p_x\}$ is closed in Y and for any $Z_n \in \mathcal{Z}_u$ $(n = 1, 2)(i)$ $\overline{Z_1 \cup Z_2} = \overline{Z_1} \cup \overline{Z_2}$ and $(ii)\overline{Z_1 \cap Z_2} = \overline{Z_1} \cap \overline{Z_2}$.

Proof. Evidently that $Z \subset \overline{Z}$. From Lemma 3.1 it follows, that if $x \in [Z]_Y$, then $Z \in p_x$ and $x \in \overline{Z}$. Hence $\overline{Z} = [Z]_Y$. The inclusion $\overline{Z_1} \cup \overline{Z_2} \subseteq \overline{Z_1 \cup Z_2}$ is obvious. Let $x \in \overline{Z_1 \cup Z_2}$. Then $Z_1 \cup Z_2 \in p_x$ and p_x is a z_u-ultrafilter on uX, converging to the point x. Since p_x is a prime z_u-filter, then either $Z_1 \in p_x$, or $Z_2 \in p_x$. So, either $x \in \overline{Z_1}$, or $x \in \overline{Z_2}$, i.e. $\overline{Z_1 \cup Z_2} \subseteq \overline{Z_1} \cup \overline{Z_2}$ and the item (i) is fulfilled.

For the item (ii) the inclusion $\overline{Z_1 \cap Z_2} \subseteq \overline{Z_1} \cap \overline{Z_2}$ is obvious. Let $x \in \overline{Z_1} \cap \overline{Z_2}$. Then $Z_1 \in p_x$, $Z_2 \in p_x$ and p_x is a z_u-ultrafilter on uX, converging to the point x. So, $Z_1 \cap Z_2 \in p_x$ and $x \in \overline{Z_1 \cap Z_2}$. The item (ii) is fulfilled. Lemma is proved. □

We continue the proof of Theorem 3.6.

Let $\alpha = \{U_i : i = 1, ..., n\} \in \mathcal{B}_p^*$ be an arbitrary finite $u-$open covering of the uniformity u_p^z (Proposition 2.1). Let $Ex_Y U_i = Y \backslash \overline{X \backslash U_i}$ $(i = 1, ..., n)$. Then $Ex_Y U_i$ is open in Y and from the equality (ii) it follows that the family $Ex_Y \alpha = \{Ex_Y U_i : i = 1, ..., n\}$ is an open covering of Y. It is easy to prove, that the finite open covering $\overline{\mathcal{B}}_p^* = \{Ex_Y \alpha : \alpha \in \mathcal{B}_p^*\}$ is a base of precompact uniformity v_p^z. By the construction $Ex_Y \alpha \wedge X = \alpha$, hence $v_p^z|_X = u_p^z$ □

Corollary 3.4. In the conditions of Theorem 3.6, for the uniformity v_p^z we have $\mathcal{Z}_{v_p^z} \wedge X = \mathcal{Z}_{u_p^z} = \mathcal{Z}_u$.

Proof. It follows from $v_p^z|_X = u_p^z$ and the item (3) of Lemma 2.4. □

Theorem 3.7. Let uX be a uniform space, X be dense in a Tychonoff space Y, and v be a uniformity on Y such that $\mathcal{Z}_v \wedge X = \mathcal{Z}_u$. The following statements are equivalent:

(1) Every $coz-$mapping f from uX into any compact uniform space νK has an extension to a $coz-$mapping $\hat{f}$ from vY into νK.
(2) uX is C_u^*-embedded in vY.
(3) Any two disjoint $u-$closed sets in uX have disjoint closures in vY.
(4) For any two $u-$closed sets Z_1 and Z_2 in uX,
$$[Z_1 \cap Z_2]_Y = [Z_1]_Y \cap [Z_2]_Y \ .$$
(5) Every point of Y is the limit of a unique z_u-ultrafilter on uX.
(6) $X \subset Y \subset \beta_u X$.
(7) $\beta_v Y = \beta_u X$.

Proof. (1) $\Rightarrow$ (2). A $u-$continuous function f in $C_u^*(X)$ is a $coz-$mapping into the compact subsets $K = [f(x)]_{\mathbb{R}}$ of $\mathbb{R}$ with respect to the uniformity $v = u_{\mathbb{R}}|_K$. Hence, item (2) is fulfilled.

(2) $\Rightarrow$ (3). It follows from Theorem 3.4.

(3) $\Rightarrow$ (4). If $x \in [Z_1]_Y \cap [Z_2]_Y$, then for every $u-$closed-set-neighborhood V of x in Y we have $x \in [V \cap Z_1]_Y$ and $x \in [V \cap Z_2]_Y$. By (3), it implies $V \cap Z_1 \cap V \cap Z_2 \neq \emptyset$, i.e. $V \cap Z_1 \cap Z_2 \neq \emptyset$. Therefore $x \in [Z_1 \cap Z_2]$. Thus $[Z_1]_Y \cap [Z_2]_Y$ is contained in $[Z_1 \cap Z_2]_Y$. The reverse inclusion is obvious.

(4) $\Rightarrow$ (5). By Lemma 3.1, each point of Y is the limit of at least one z_u-ultrafilter. Distinct z_u-ultrafilters contain disjoint $u-$closed sets (Theorem 3.2 (c)) and by (4) it implies that a point x cannot belong to the closures of both these $u-$closed sets. Hence, the two distinct z_u-ultrafilters cannot both converge to x.

(5) $\Rightarrow$ (1). Given $x \in Y$, let p_x denote the unique z_u-ultrafilter on uX with the limit x. As in Lemma 2.2, we write $f^\sharp(p_x) = \{E \in \mathcal{Z}_\nu : f^{-1}(E) \in p_x\}$. This is a prime $z_\nu-$filter on a compact uniform space νK, and so it has a cluster point. Therefore, by Theorem 3.2, $f^\sharp(p_x)$ has a limit in νK. Denote this limit by $\{y\} = \cap\{f^\sharp(\mathcal{F})\}$. It means that it is determined a mapping $\hat{f}$ from vY into νK. In case $x \in X$, we have $\{x\} = \cap p_x$, so that $y = \hat{f}(x) = f(x) = \cap f^\sharp(p_x)$. Therefore $\hat{f}$ agrees with f on X. As $f^{-1}(F) \in \mathcal{Z}_u$ for all $\nu-$closed sets $F \in \mathcal{Z}_\nu$, then for a mapping $\hat{f} : Y \to \nu K$ the equality $\overline{f^{-1}(F)} = \hat{f}^{-1}(F)$ holds for all $F \in \mathcal{Z}_\nu$, where $\overline{f^{-1}(F)} = \{x \in Y : f^{-1}(F) \in p_x\}$ (as in the proof of Theorem 3.6). Then for any finite cozero-covering $\beta = \{V_i : i = 1, 2, ..., n\} \in \nu$ of the compact K, the covering $\hat{f}^{-1}(\beta) = \{\hat{f}^{-1}(V_i) : i = 1, 2, ..., n\}$ is an open covering of Y, as, by Theorem 3.6, $\hat{f}^{-1}(V_i) = Y \setminus \overline{f^{-1}(K \setminus V_i)}$ $(i = 1, 2, ..., n)$ and $\hat{f}^{-1}(\beta) \in v_p^z$. Hence $\hat{f} : v_p^z Y \to \nu K$ is a uniformly continuous mapping. By Corollary 3.4, $\mathcal{Z}_{v_p^z} \wedge X = \mathcal{Z}_{u_p^z} = \mathcal{Z}_u$. We note that $\hat{f}^{-1}(F) \in \mathcal{Z}_{u_p^z}$ for any $F \in \mathcal{Z}_\nu$. Evidently, $\hat{f}^{-1}(F) \cap X = \overline{f^{-1}(F)} \cap X = f^{-1}(F)$ and $f^{-1}(F) \in \mathcal{Z}_v \wedge X$. Then there exist $v-$closed sets $Z_n \in \mathcal{Z}_v$ $(n \in \mathbb{N})$ such that $Int(Z_n) \neq \emptyset$ and

$f^{-1}(F) = \cap_{n\in\mathbb{N}}\{Z_n \cap X\}$. We have $\hat{f}^{-1}(F) = \overline{f^{-1}(F)} = \overline{\cap_{n\in\mathbb{N}}\{Z_n \cap X\}} = \cap_{n\in\mathbb{N}} Z_n$ (it follows from the proof of Theorem 3.6). Thus, $\hat{f}^{-1}(F)$ is $v-$closed set, i.e. the mapping $\hat{f} : vY \to \nu K$ is $v-$continuous.

(5) $\Rightarrow$ (7). By Theorem 3.6, a completion of Y, with respect to the uniformity v_p^z, is the Samuel compactification $s_{v_p^z}Y$ of the uniform space v_p^zY and $v_p^z|_X = u_p^z$. Since X is dense in Y, then $s_{v_p^z}Y = \beta_u X$. From (7) of Theorem 2.2 and (5) it follows that each point of the compactification $s_{v_p^z}Y$ is the limit of a unique z_u-ultrafilter on uX, hence by Corollary 3.4, each point of $s_{v_p^z}Y$ is the limit of a unique z_v-ultrafilter on vY. So, by (7) of Theorem 2.2, we have $s_{v_p^z}Y = \beta_v Y = \beta_u X$.

(7) $\Rightarrow$ (6). $X \subset Y \subset \beta_v Y = \beta_u X$.

(6) $\Rightarrow$ (2). The uniform space uX is C_u^*-embedded in the compactification $\beta_u X$. By (1) of Theorem 2.1, (2) of Theorem 2.2, Theorem 3.6 and Corollary 3.2, it follows, that uX is C_u^*-embedded in the uniform space vY. □

Corollary 3.5. Let uX be a dense uniform subspace of the uniform space vY. The following statements are equivalent:

(1) Every *coz*$-$mapping f from uX into any compact uniform space νK has an extension to a *coz*$-$mapping $\hat{f}$ from vY into νK.
(2) uX is C_u^*-embedded in vY.
(3) Any two disjoint $u-$closed sets in uX have disjoint closures in vY.
(4) For any two $u-$closed sets Z_1 and Z_2 in uX,
$$[Z_1 \cap Z_2]_Y = [Z_1]_Y \cap [Z_2]_Y \ .$$
(5) Every point of Y is the limit of a unique z_u-ultrafilter on uX.
(6) $X \subset Y \subset \beta_u X$.
(7) $\beta_v Y = \beta_u X$.

Proof. It immediately follows from Theorem 3.7, since $u = v|_X$, hence $\mathcal{Z}_v \wedge X = \mathcal{Z}_u$. □

Theorem 3.8. Every uniform space uX has a $\beta-$like compactification $\beta_u X$ with the next equivalent properties:

(I) Every *coz*$-$mapping f from uX into any compact space νK has a continuous extension $\beta_u f$ from $\beta_u X$ into K.
(II) uX is C_u^*-embedded in $\beta_u X$.
(III) Any two disjoint $u-$closed sets in uX have disjoint closures in $\beta_u X$.
(IV) For any two $u-$closed sets Z_1 and Z_2 in uX,
$$[Z_1 \cap Z_2]_{\beta_u X} = [Z_1]_{\beta_u X} \cap [Z_2]_{\beta_u X} \ .$$
(V) Distinct z_u-ultrafilters on uX have distinct limits in $\beta_u X$.

The compactification $\beta_u X$ is unique in the following sense: if a compactification Y of uX satisfies any of listed conditions, then there exists a homeomorphism of $\beta_u X$ onto Y that leaves X pointwise fixed.

Proof. By Theorem 3.7, if a compactification Y satisfies any of (I) - (IV), it satisfies all of them. By (I), the identity mapping on uX (which is a *coz*$-$mapping into the compact uniform space vY) has a $\beta-$like extension from $\beta_u X$ into vY; similarly, it has an extension from vY into $\beta_u X$ (by Corollary 2.1). It follows that these extensions are homeomorphisms onto [10, 2.1.9, 3.5.4]. □

Proposition 3.2. The next statements are equivalent:

(a) In a uniform space uX any two disjoint closed sets, one of which is compact, are $u-$separated.
(b) In a uniform space uX every $G_\delta-$set containing a compact set K, contains a $u-$closed set containing K.
(c) Every compact uniform subspace νK of a uniform space uX is $C_\nu-$embedded.

Proof. (a) Let F and F' be disjoint closed sets in uX with F is compact. For each $x \in F$, choose disjoint $u-$closed sets Z_x and Z'_x, with Z_x is a $u-$closed-sets-neighborhood of x and $Z'_x \supset F'$. The covering $\{Z_x : x \in F\}$ of the compact set F has a finite subcovering $\{Z_{x_1}, ..., Z_{x_n}\}$. Then F and F' are contained in the disjoint $u-$closed sets $Z_{x_1} \cup ... \cup Z_{x_n}$ and $Z'_{x_1} \cap ... \cap Z'_{x_n}$, respectively. Hence, by Theorem 3.1, F and F' are $u-$separated.

(b) A $G_\delta-$set G has the form $\cap_{n\in\mathbb{N}} U_n$, where each U_n is open in uX. If $G \supset K$, then K is $u-$separated from $X \backslash U_n$, by item (a), and so, by Corollary 3.1, there is a $u-$closed set F_n satisfying $K \subset F_n \subset U_n$. Then $K \subset \cap_{n\in\mathbb{N}} F_n \subset G$ and $\cap_{n\in\mathbb{N}} F_n$, as a countable intersection of $u-$closed sets is a $u-$closed set.

(c) Let νK be a compact uniform subspace of a uniform space uX. If F and F' are $\nu-$separated in νK, then F and F' have disjoint closures in K. As these closures are compact, they are, by (a), $u-$separated in uX. By Theorem 3.4, compact νK is $C^*_\nu-$embedded in uX. By (b), the compact set K is $u-$separated from every $u-$closed set disjoint from it. Hence the compact uniform subspace νK is $C_\nu-$embedded in uX. □

Proposition 3.3. Let $u'S$ be a uniform subspace of uX. Then
(a) $u'S$ is $C^*_{u'}-$embedded in uX if and only if it is $C^*_{u'}-$embedded in $\beta_u X$.
(b) $u'S$ is $C^*_{u'}-$embedded in uX if and only if $[S]_{\beta_u X} = \beta_{u'} S$.

Proof. (*a*) It is obvious.

(*b*) By (c) of Proposition 3.2, the compact uniform subspace $K = [S]_{\beta_u X}$ of the compactum $\beta_u X$ is $C^*_\nu-$embedded in $\beta_u X$, where ν is a uniformity on K, induced by the unique uniformity of the compactification $\beta_u X$. So, the conditions of (b) hold if and only if the uniform space $u'S$ is $C^*_{u'}-$embedded in $\beta_{u'} S$ and the compactum $K = [S]_{\beta_u X}$ satisfies (2) of Theorem 3.7 and is a compactification of $u'S$, in which $u'S$ is $C^*_{u'}-$embedded. □

Remark 3.2. In [8] there is an example of a uniform space uX such that $\beta_u X \neq \beta X$. Then uX is C^*_u-embedded, but it is not C^*-embedded in the compactification $\beta_u X$, because if uX is C^*-embedded in $\beta_u X$, then $\beta_u X = \beta X$. A contradiction.

4. C_u- EMBEDDING IN REALCOMPACTIFICATIONS

Definition 4.1. [12] A mapping $f : uX \to vY$ is said to be a *coz−homeomorphism*, if f is a *coz−* mapping of uX onto vY in a one-to-one way, and the inverse mapping $f^{-1} : vY \to uX$ is a *coz*−mapping. A two uniform spaces uX and $vY$are *coz−homeomorphic* if there exists a *coz−* homeomorphism of uX onto vY.

Definition 4.2. A uniform space uX is said to be *uniformly realcompact* if it is *coz*−homeomorphic to a closed uniform subspace of a power of $\mathbb{R}$.

Remark 4.1. By analogue with [13], an ideal $I \subset C_u(X)$ is said to be a *fixed*, if $\cap \mathbf{Z}(I) = \cap\{\mathbf{Z}(f) : f \in I\} \neq \emptyset$, and if $\mathbf{Z}(I)$ is a countably centered z_u-ultrafilter, then a maximal ideal I is said to be a *real* ideal.

Theorem 4.1. For uniform space uX the following conditions are equivalent:

(1) uX is uniformly realcompact;

(2) X is complete with respect to the uniformity u^z_ω;
(3) $uX = \upsilon_u X$;
(4) each countably centered z_u−ultrafilter is convergent;
(5) each point in X is the limit of a unique countably centered z_u−ultrafilter on uX.
(6) every real maximal ideal in $C_u(X)$ is fixed.

Proof. (1) ⇒ (2) Let $i : uX \to \mathbb{R}^\tau$ be a coz−homeomorphism of the uniform space uX onto a closed uniform subspace $X' = i(X) \subset \mathbb{R}^\tau$ with the uniformity $u' = u^\tau_{\mathbb{R}}|_{i(X)}$, where $\mathbb{R}^\tau(u^\tau_{\mathbb{R}})$ is a power of $\mathbb{R}(u_{\mathbb{R}})$. The uniform space $u^\tau_{\mathbb{R}}\mathbb{R}^\tau$ is $\aleph_0$−bounded and complete [3], hence $u'X'$ is also $\aleph_0$−bounded and complete [3]. Then X' is complete with respect to the uniformity u'^z_ω (Proposition 2.1). From (1') of Lemma 2.1 it follows that the uniform spaces $u^z_\omega X$ and $u'^z_\omega X'$ are uniformly homeomorphic, so X is complete with respect to the uniformity u^z_ω (Proposition 2.1).

(2) ⇔ (3) It follows from items (1), (2) of Theorem 2.3.

(3) ⇔ (4) ⇔ (5). It follows from items (1), (8) of Theorem 2.4.

(5) ⇔ (6). It is obvious (Remark 4.1).

(2) ⇒ (1). Let $|C_u(X)| = \tau$. By Lemma 2.1 (2), $C_u(X) = U(u^z_\omega X)$, hence the uniform space $u^z_\omega X$ is uniformly homeomorphically embedded into $\mathbb{R}^\tau$, i.e. the uniform space uX is coz−homeomorphically embedded into $\mathbb{R}^\tau$. From (2) it follows that uX is coz−homeomorphic to a closed uniform subspace of $u^\tau_{\mathbb{R}}\mathbb{R}^\tau$. □

Lemma 4.1. [21] If $p \subset \mathcal{Z}(X)$ is a filter closed under countable intersections and $\cap p = \emptyset$, then on a Tychonoff space X there exists a closed set base, which is separating, nest-generated intersection ring and there exists a uniformity u such, that $p \in \upsilon_u X$.
Proof. [21, Lemma 3.5]. We put $\mathcal{F} = \{Z \in \mathcal{Z}(X) : Z \in p \text{ or } Z \cap P = \emptyset \text{ for some } P \in p\}$. Then $\mathcal{F}$ is separating, nest-generated intersection ring and the Wallman compactification $\omega(X, \mathcal{F})$ is a β−like [21] compactification. All countable coverings from the family $C\mathcal{F} = \{X \backslash Z : Z \in \mathcal{F}\}$ are the uniformity u on X. Therefore $\omega(X, \mathcal{F}) = \beta_u X$, $\upsilon(X, \mathcal{F}) = \upsilon_u X$ and p is a free countably centered z_u−ultrafilter on the uniform space uX, i.e. $p \in \upsilon_u X$. □

Corollary 4.1. If X is a realcompact and non-Lindelöf space, then there exists a uniformity u on X such that uX is not uniformly realcompact. The uniform space uX is C_u−embedded, but it is not C−embedded in $\upsilon_u X$.

Proof. If X is realcompact and non-Lindelöf, then there is a filter $p \subset \mathcal{Z}(X)$, which is closed under countable intersections, and $\cap p = \emptyset$ [10, 3.8.3]. By Lemma 4.1, on X there exists a uniformity u such, that $X \neq \upsilon_u X$, i.e. the uniform space uX is not uniformly realcompact. Evidently, uX is C_u−embedded in $\upsilon_u X$. If uX is C−embedded in $\upsilon_u X$, then $\upsilon X = X = \upsilon_u X$, but $X \neq \upsilon_u X$, and we have a contradiction. □

The next theorem characterizes the Tychonoff Lindelöf spaces by means of uniform structures.

Theorem 4.2. A Tychonoff space X is Lindelöf if and only if uX is uniformly realcompact for any uniformity u on X.

Proof. If a Tychonoff space X is Lindelöf, then evidently uX is uniformly realcompact ([10, 3.8.3], item (2) of Theorem 4.1).

Let uX be uniformly realcompact for any uniformity u on X. Suppose that X is a non-Lindelöf space. Then on X there is a countably centered z−filter $p \subset \mathcal{Z}(X)$ such that $\cap p = \emptyset$ [10, 3.8.3]. Then, by Lemma 4.1, there exists a uniformity u on X such that $X \neq \upsilon_u X$, i.e. the uniform space uX is not uniformly realcompact. A contradiction. □

Corollary 4.2. Every open uniform subspace of the $\aleph_0-$bounded metrizable uniform space is uniformly realcompact.

Proof. Every $\aleph_0-$bounded metrizable uniform space possesses a countable base [3]. Hence it is hereditary Lindelöf, i.e. any open subspace is Lindelöf [10, 3.8.A] and it is uniformly realcompact with respect to each uniformity on it. □

Proposition 4.1. A closed uniform subspace of a uniformly realcompact space is uniformly realcompact.

Proof. Let vY be a closed uniform subspace of the uniformly realcompact space uX, where $v = u|_Y$. A space X is complete under the uniformity u_ω^z (Proposition 2.1 and item (2) of Theorem 4.4), hence vY is complete with respect to the uniformity $u_\omega^z|_X$. As $u_\omega^z|_X \subset v_\omega^z$ (Proposition 2.1), vY is complete with respect to the uniformity v_ω^z and the uniform space vY is uniformly realcompact. □

Proposition 4.2. A product of any collection of uniformly realcompact spaces is uniformly realcompact if and only if every factor is uniformly realcompact.

Proof. Let $\{u_tX_t : t \in T\}$ be an arbitrary collection of the uniformly realcompact spaces, i.e. u_tX_t is complete under the uniformity $u_{t,\omega}^z$ (Proposition 2.1) for any $t \in T$. Let $X = \prod\{X_t : t \in T\}$, $u = \prod\{u_t : t \in T\}$ and $v = \prod\{u_{t,\omega}^z : t \in T\}$. Then the uniform space uX is complete with respect to the uniformity v. Evidently, $v \subset u_\omega^z$ (Proposition 2.1). So, uX is complete with respect to the uniformity u_ω^z and uX is a uniformly realcompact space.

The proof of the second part follows from Proposition 4.9. □

From Propositions 4.1. and 4.2. the next statement immediately follows.

Corollary 4.2. A limit of an inverse system consisting of uniformly realcompact spaces and "short" projections, being $coz-$mappings, is uniformly realcompact.

Corollary 4.3. Let $\{u_tX_t : t \in T\}$ be a collection of uniformly realcompact uniform subspaces of the uniformly realcompact space uX, i.e. $u_t = u|_{X_t}$ for any $t \in T$. Then the intersection $\cap\{X_t : t \in T\} = Y$, equipped by the uniformity $v = u|_Y$, is uniformly realcompact.

Proof. Let $X' = \prod\{X_t : t \in T\}$ and $u' = \prod\{u_t : t \in T\}$. Then the uniform space $u'X'$ is uniformly realcompact (by Proposition 4.2) and it is a uniform subspace of u^TX^T, where T is a power of uX. The diagonal Δ of the power X^T is a closed subspace. Evidently, the uniform space vY is uniformly homeomorphic to the closed in X' uniform subspace $\Delta \cap X'$, equipped by the uniformity $u'|_{\Delta\cap X'}$, which is uniformly realcompact (by Proposition 4.1). □

Theorem 4.3. Let uX be a uniform space, X be dense in a Tychonoff space Y, and v be a uniformity on Y such that $\mathcal{Z}_v \wedge X = \mathcal{Z}_u$, and every point of Y be a limit of a unique countably centered z_u-ultrafilter on uX. Then there exists a $\aleph_0-$bounded uniformity v_ω^z on Y such that $v_\omega^z|_X = u_\omega^z$

Proof. For any point $x \in Y$, p_x is a unique countably centered z_u-ultrafilter on uX, converging to the point x. Let $Z \in \mathcal{Z}_u$ be an arbitrary member.

Lemma 4.2. In the conditions of Theorem 4.3, if $Z \in \mathcal{Z}_u$, then the set $\overline{Z} = \{x \in Y : Z \in p_x\}$ is closed in Y and for any collection of $u-$closed sets $\{Z_n\}_{n\in\mathbb{N}}$ in uX the equality $\overline{\cap_{n\in\mathbb{N}}Z_n} = \cap_{n\in\mathbb{N}}\overline{Z_n}$ is fulfilled.

Proof. From Lemma 3.1, it follows that if $x \in [Z]_Y$, then $Z \in p_x$ and $x \in \overline{Z}$, i.e. $[Z]_Y \subseteq \overline{Z}$. On the other hand, $\overline{Z} \subseteq [Z]_Y$, which means $\overline{Z} = [Z]_Y$. It is clear, that $\overline{\cap_{n\in\mathbb{N}}Z_n} \subseteq \cap_{n\in\mathbb{N}}\overline{Z_n}$. Then

$x \in \cap_{n\in\mathbb{N}}\overline{Z_n}$, i.e. $x \in \overline{Z_n}$ for all $n \in \mathbb{N}$. Then $Z_n \in p_x$ for all $n \in \mathbb{N}$ and p_x is a countably centered z_u-ultrafilter on uX. Therefore $\cap_{n\in\mathbb{N}}Z_n$ is $u-$closed and $\cap_{n\in\mathbb{N}}Z_n \in p_x$, i.e. $x \in \overline{\cap_{n\in\mathbb{N}}Z_n}$. So, $\cap_{n\in\mathbb{N}}\overline{Z_n} \subseteq \overline{\cap_{n\in\mathbb{N}}Z_n}$. The lemma is proved. □

We continue the proof of Theorem 4.3.

Let $\alpha = \{U_i\}_{i\in\mathbb{N}} \in \mathcal{B}^*_\omega$ be an arbitrary countable $u-$open covering of the uniformity u^z_ω (Proposition 2.2).
Let $Ex_YU_i = Y\backslash\overline{X\backslash U_i}$ $(i \in \mathbb{N})$. Then Ex_YU_i is open in Y and from the equality (ii) it follows, that the family $Ex_Y\alpha = \{Ex_YU_i\}_{i\in\mathbb{N}}$ is an open covering of Y. It is easy to check, that the collection $\hat{\mathcal{B}}^*_\omega = \{Ex_Y\alpha : \alpha \in \mathcal{B}^*_\omega\}$ of countable open coverings is a base of $\aleph_0-$bounded uniformity v^z_ω. By the construction $Ex_Y\alpha \wedge X = \alpha$, hence, $v^z_\omega|_X = u^z_\omega$. □

Corollary 4.3. In the conditions of Theorem 4.3, for the uniformity v^z_ω we have $\mathcal{Z}_{v^z_\omega} \wedge X = \mathcal{Z}_{u^z_\omega} = \mathcal{Z}_u$.

Proof. It follows from $v^z_\omega|_X = u^z_\omega$ and the item (3) of Lemma 2.1. □

Theorem 4.4. Let uX be a uniform space, X be dense in a Tychonoff space Y, and v be a uniformity on Y such that $\mathcal{Z}_v \wedge X = \mathcal{Z}_u$. The following statements are equivalent:

(1) Every $coz-$mapping f from uX into any uniformly realcompact uniform space νR has a $coz-$extension to a $coz-$mapping $\hat{f}$ from vY into νR.
(2) uX is C_u-embedded in vY.
(3) If a countable $u-$closed sets family in uX has empty intersection, then their closures in vY have empty intersection.
(4) For any countable family of $u-$closed sets $\{Z_n\}_{n\in\mathbb{N}}$ in uX,
$$[\cap_{n\in\mathbb{N}}Z_n]_Y = \cap_{n\in\mathbb{N}}[Z_n]_Y\ .$$
(5) Every point of vY is the limit of a unique countably centered z_u-ultrafilter on uX.
(6) $X \subset Y \subset \upsilon_uX$.
(7) $\upsilon_vY = \upsilon_uX$.

Proof. (1) ⇒ (2) It is obvious, as $u_\mathbb{R}\mathbb{R}$ is uniformly realcompact.

(2) ⇒ (3) Let $\cap_{n\in\mathbb{N}}Z_n = \emptyset$ and $Z_n \in \mathcal{Z}_u$, $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, $Z_n = f_n^{-1}(0)$, $f_n \in C_u(X)$. Because uX is dense and C_u-embedded in vY, then the functions f_n uniquely can be extended to a functions $\hat{f}_n \in C_v(Y), n \in \mathbb{N}$. Evidently, $[Z_n] \subseteq \hat{f}_n^{-1}(0)$. We show that the family $\hat{\alpha} = \{Y \setminus \hat{f}_n^{-1}(0) : n \in \mathbb{N}\}$ is a covering of Y. Then the family $\alpha = \{Y \setminus [Z_n]_{n\in\mathbb{N}}\}$, a fortiori, will be a covering of Y. Suppose, that $y \in Y \setminus \cup\hat{\alpha}$, i.e. $y \in \cap_{n\in\mathbb{N}}\hat{f}_n^{-1}(0)$. Let p_y be a countably centered z_v-ultrafilter such that $\cap p_y = \{y\}$. Then $\hat{f}_n^{-1}(0) \in p_y$ and $p_y \wedge X$ is a countably centered $u-$closed sets family. As X is dense in Y, $p_y \wedge X \neq \emptyset$. Let p be a countably centered z_u-ultrafilter containing $p_y \wedge X$. Then $\hat{f}_n^{-1}(0) \cap X = \mathcal{Z}_u \in p, n \in \mathbb{N}$ and hence $\cap_{n\in\mathbb{N}}Z_n \neq \emptyset$. Contradiction. Otherwise, the family α is a covering of Y, therefore, $\cap_{n\in\mathbb{N}}[Z_n] = \emptyset$.

(3) ⇒ (4) $[\cap_{n\in\mathbb{N}}Z_n]_Y \subseteq \cap_{n\in\mathbb{N}}[Z_n]_Y$ is obvious. Conversely, let $x \in \cap_{n\in\mathbb{N}}[Z_n]_Y$. Then for any $v-$closed neighborhood of x we have $x \in [V \cap Z_n]_Y, n \in \mathbb{N}$ and $x \in \cap_{n\in\mathbb{N}}[V \cap Z_n]_Y$. By (3) we have $\cap_{n\in\mathbb{N}}(V \cap Z_n) \neq \emptyset$, i.e. $V \cap (\cap_{n\in\mathbb{N}}Z_n) \neq \emptyset$. So, $x \in [\cap_{n\in\mathbb{N}}Z_n]$, and (4) $(\cap_{n\in\mathbb{N}}Z_n)$ holds.

(4) ⇒ (5) It is obvious.

(5) ⇒ (1) Let $x \in Y$. Let p_x denote the unique countably centered z_u-ultrafilter on uX with limit x. As in Lemma 2.2, we write $f^\sharp(p_x) = \{E \in \mathcal{Z}_\nu : f^{-1}(E) \in p_x\}$. This is a countably centered prime $z_\nu-$filter on the uniformly realcompact space νR. Then, by Corollary 2.2, $f^\sharp(p_x)$ is contained in the unique countably centered $z_\nu-$ultrafilter p^x. So, by Theorem 3.2.,$f^\sharp(p_x)$ and p^x are converging to the same limit. Denote this limit by $\{y\} = \cap p^x = f^\sharp(p_x)$.

It means that determines a mapping $\tilde{f}$ from vY into νR. In case $x \in X$, we have $\{x\} = \cap p_x$, so that $y = \tilde{f}(x) = f(x) = \cap f^\sharp(p_x)$. Therefore $\tilde{f}$ agrees with f on X. As $f^{-1}(F) \in \mathcal{Z}_u$ for all $F \in \mathcal{Z}_\nu$, then for a mapping $\tilde{f} : Y \to \nu R$ the equality $\overline{f^{-1}(F)} = \tilde{f}^{-1}(F)$ is fulfilled for every $F \in \mathcal{Z}_\nu$, where $\overline{f^{-1}(F)} = \{x \in Y : f^{-1}(F) \in p_x\}$ (as in the proof of Theorem 3.6). Then for any countable $\nu-$open covering $\beta = \{U_i\}_{i\in\mathbb{N}} \in \nu_\omega^z$ of the uniformly realcompact νR the covering $\hat{f}^{-1}(\beta) = \{\hat{f}^{-1}(U_i)\}_{i\in\mathbb{N}}$ is open covering of Y, as, by the Theorem 4.3, $\tilde{f}^{-1}(U_i) = Y \setminus \overline{f^{-1}(R \setminus U_i)}$ $(i \in \mathbb{N})$ and $\tilde{f}^{-1}(\beta) \in v_\omega^z$. Hence $\hat{f} : v_\omega^z Y \to \nu R$ is uniformly continuous mapping. By Corollary 3.4, $\mathcal{Z}_{v_\omega^z} \wedge X = \mathcal{Z}_{u_\omega^z} = \mathcal{Z}_u$. We note that $\tilde{f}^{-1}(F) \in \mathcal{Z}_{u_\omega^z}$ for any $F \in \mathcal{Z}_\nu$. Evidently, $\tilde{f}^{-1}(F) \cap X = \overline{f^{-1}(F)} \cap X = f^{-1}(F)$ and $f^{-1}(F) \in \mathcal{Z}_v \wedge X$. Then there exist $v-$closed sets $Z_n \in \mathcal{Z}_v$ $(n \in \mathbb{N})$ such that $Int(Z_n) \neq \emptyset$ and $f^{-1}(F) = \cap_{n\in\mathbb{N}}\{Z_n \cap X\}$. We have $\tilde{f}^{-1}(F) = \overline{f^{-1}(F)} = \overline{\cap_{n\in\mathbb{N}}\{Z_n \cap X\}} = \cap_{n\in\mathbb{N}} Z_n$ (it follows from Theorem 4.3). Thus, $\tilde{f}^{-1}(F)$ is a $v-$closed set, i.e. the mapping $\tilde{f} : vY \to \nu R$ is $v-$continuous.

(5) ⇒ (7) By Theorem 4.3, a completion $\tilde{v}_\omega^z \tilde{Y}$ of the space Y, with respect to the uniformity v_ω^z, coincides with the Wallman realcompactification $\upsilon_u X$. From the items (1), (5) of Theorem 2.3 and item (8) of Theorem 2.4, it follows that each point of $\tilde{v}_\omega^z \tilde{Y}$ is the limit of a unique countably centered z_u-ultrafilter on uX, hence, by Corollary 4.3, each point of $\tilde{v}_\omega^z \tilde{Y}$ is the limit of a unique countably centered z_v-ultrafilter on vY. By the item (8) of Theorem 2.104, we have $\tilde{v}_\omega^z \tilde{Y} = \upsilon_v Y = \upsilon_u X$.

(7) ⇒ (6) $X \subset Y \subset \upsilon_v Y = \upsilon_u X$.

(6) ⇒ (2) The uniform space uX is C_u-embedded in the Wallman realcompactification $\upsilon_u X$. By the item (1) of Theorem 2.3, item (5) of Theorem 2.4, Theorem 4.3 and Corollary 3.2, it follows that uX is C_u-embedded in the uniform space vY. □

Corollary 4.4. Let uX be a dense uniform subspace of a uniform space vY. The following statements are equivalent:

(1) Every $coz-$mapping f from uX into any uniformly realcompact uniform space νR has a $coz-$extension to a $coz-$mapping $\hat{f}$ from vY into νR.
(2) uX is C_u-embedded in vY.
(3) If a countable $u-$closed sets family in uX has empty intersection, then their closures in vY have empty intersection.
(4) For any countable family of $u-$closed sets $\{Z_n\}_{n\in\mathbb{N}}$ in uX,
$$[\cap_{n\in\mathbb{N}} Z_n]_Y = \cap_{n\in\mathbb{N}} [Z_n]_Y.$$
(5) Every point of vY is the limit of a unique countably centered z_u-ultrafilter on uX.
(6) $X \subset Y \subset \upsilon_u X$.
(7) $\upsilon_v Y = \upsilon_u X$.

Proof. It immediately follows from Theorem 4.4, since $u = v|_X$, hence $\mathcal{Z}_v \wedge X = \mathcal{Z}_u$. □

Theorem 4.5. Every uniform space uX has the Wallman realcompactification $\upsilon_u X$, contained in a $\beta-$like compactification $\beta_u X$ with the next equivalent properties:

(I) Every $coz-$mapping f from uX into any uniformly realcompact space νR has a continuous $coz-$extension $\tilde{f}$ from $\upsilon_u X$ into vY.
(II) uX is C_u-embedded in $\upsilon_u X$.
(III) If a countable family of $u-$closed sets in uX has empty intersection, then their closures in $\upsilon_u X$ have empty intersection.
(IV) For any countable family of $u-$closed sets $\{Z_n\}_{n\in\mathbb{N}}$ in uX,
$$\cap_{n\in\mathbb{N}} [Z_n]_{\upsilon_u X} = [\cap_{n\in\mathbb{N}} Z_n]_{\upsilon_u X} .$$

(V) Every point of $\upsilon_u X$ is the limit of a unique countably centered z_u-ultrafilter.

The Wallman realcompactification $\upsilon_u X$ is unique in the following sense: if a uniform space vY is a realcompactification of uX satisfies any one of listed conditions, then there exists a $coz-$homeomorphism of $\upsilon_u X$ onto vY that leaves X pointwise fixed.

Proof. It is analogically to Theorem 3.8, for the case $\beta-$like compactification $\beta_u X$. □

5. Acknowledgments

The authors thank the referee for a number of helpful suggestions.

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