TWMS J. Pure Appl. Math. V.9, N.2, 2018, pp.190-206

NORMALIZED WRIGHT FUNCTIONS WITH NEGATIVE COEFFICIENTS AND SOME OF THEIR INTEGRAL TRANSFORMS

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Abstract. The purpose of the present paper is to investigate some characterization for the normalized Wright functions to be in the subclass $T(\alpha,\beta)(\alpha \in [0,1), \beta \in [0,1))$ of analytic functions in the open unit disk. Several sufficient conditions were obtained for the parameters of the normalized form of the Wright functions to be in this class. Some geometric properties of integral transforms involving normalized Wright functions are also studied. The results obtained here are new and their usefulness is depicted by deducing several interesting corollaries and examples.

Keywords: Wright function, analytic functions, starlike functions, convex functions, integral transforms.

AMS Subject Classification: 30C45, 30D20, 33E20, 44A20.

1. Introduction

It is well known that the special functions play an important role in the geometric function theory. It is also well known that the application area of the special functions is not limited to the theory of geometric functions. The special functions have wide range of applications in many problems as well as in other branches of mathematics and applied sciences.

The Wright function $W_{\lambda,\mu}(z)$ is defined by the series

$$W_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\lambda n + \mu)} \frac{z^n}{n!}, \ \lambda > -1, \ \mu, z \in \mathbb{C}. \tag{1}$$

This series is absolutely convergent in $\mathbb{C}$, when $\lambda > -1$ and absolutely convergent in open unit disk for $\lambda = -1$. Furthermore, for $\lambda > -1$, the Wright function is an entire function. The Wright function $W_{\lambda,\mu}(z)$ was introduced by Wright in [22], and has appeared for the first time in the case $\lambda > 0$ in connection with his investigations in the asymptotic theory of the partitions. Later on, many other applications have been found, first of all, in the Mikusinski operational calculus and in the theory of integral transforms of Hankel type. Furthermore, extending the methods of Lie groups in the partial differential equations to the partial differential equations of the fractional order, it was shown that some of the group-invariant solutions of these equations can be given in terms of the Wright function.

Recently, this function has appeared in the solution of the partial differential equations of the fractional order, it was found that the corresponding Green functions can be represented in terms of the Wright function (see [14], [19]). There are papers devoted to the applications

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Manuscript received November 2017 .

of the Wright function in the partial differential equation of the fractional order extending the classical diffusion and wave equations. In [11] Mainardi has obtained the result for a fractional diffusion wave equation in terms of the fractional Green function involving the Wright function. The scale-variant solutions of the some partial differential equations of the fractional order were obtained in terms of the special cases of the generalized Wright function by Buckwar and Luchko [5] and Luchko and Gorenflo [10].

If λ is a positive rational number, then the Wright function $W_{\lambda,\mu}(z)$ can be represented in terms of the more familiar generalized hypergeometric function (see [8, Section 2.1]). In particular, when $\lambda = 1$ and $\mu = \nu + 1$, the functions $W_{1,\nu+1}(-z^2/4)$ are expressed in terms of the Bessel functions given as follows:

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu W_{1,\nu+1}\left(\frac{-z^2}{4}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(z/2)^{2n+\nu}}{\Gamma(n+\nu+1)}.$$

Furthermore, the function $W_{\lambda,\nu+1}(-z) \equiv J_\nu^\lambda(z)$ $(\lambda > 0,\ \nu > -1)$ is known as the generalized Bessel function (misnamed also as the Bessel-Maitland function). Also, the Wright function generalizes various simple functions like the Array function, Wittaker function, (Wright-type) entire auxiliary functions (see for details [8]).

Several researchers studied classes of analytic functions involving special functions $F \subset A$, to find different conditions such that the members of F have certain geometric properties such as starlikeness or convexity in the open unit disk. There is an extensive literature dealing with geometric properties of different types of the hypergeometric functions, especially for generalized Gaussian, Kummer and generalized hypergeometric, Mittag-Leffler type, Bessel functions and harmonic preinvex, and harmonic univalent functions with varying arguments defined by using Salagean integral operator ([4] - [18], [21]).

2. Preliminaries

Let T be the class of analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ functions $f(z)$, normalized by $f(0) = 0 = f'(0) - 1$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n,\ a_n \geq 0. \tag{2}$$

We denote by $TS^*(\alpha)$ and $TC(\alpha)$ the subclasses of T consisting of the functions which are, respectively, starlike or convex of order α $(\alpha \in [0,1))$ in the open unit disk U. From the definition, we have (see for details [6]-[20])

$$TS^*(\alpha) = \left\{ f \in T : \Re(\frac{zf'(z)}{f(z)}) > \alpha,\ z \in U \right\},\ \alpha \in [0,1) \tag{3}$$

and

$$TC(\alpha) = \left\{ f \in T : \Re(1 + \frac{zf''(z)}{f'(z)}) > \alpha,\ z \in U \right\}, \alpha \in [0,1). \tag{4}$$

An interesting unification of the function classes $TS^*(\alpha)$ and $TC(\alpha)$ is provided by the class $T(\alpha, \beta)$ of functions $f \in T$, which also satisfies the following condition:

$$\Re\left\{ \frac{zf'(z) + \beta z^2 f''(z)}{\beta z f'(z) + (1-\beta) f(z)} \right\} > \alpha,\ z \in U,\ 0 \leq \alpha < 1,\ 0 \leq \beta \leq 1.$$

Thus,

$$\begin{aligned} T(\alpha,\beta) &= \left\{ f \in T : \Re\left(\frac{zf'(z)+\beta z^2 f''(z)}{\beta z f'(z)+(1-\beta)f(z)} \right) > \alpha, z \in U \right\}, \\ 0 &\leq \alpha < 1, \ 0 \leq \beta \leq 1. \end{aligned} \tag{5}$$

The class $T(\alpha,\beta)$ was investigated by Altintaş et al. [2] and [3] (in a more general way $T_n(p,\alpha,\beta)$) and (subsequently) by Irmak et al. [9]. In particular, the class $T_n(1,\alpha,\beta)$ was considered earlier by Altintaş [1].

In special case for $\beta = 0$ and $\beta = 1$, we have

$$T(\alpha,0) = TS^*(\alpha) \text{ and } T(\alpha,1) = TC(\alpha) \tag{6}$$

in terms of the simpler classes $TS^*(\alpha)$ and $TC(\alpha)$, defined by (3) and (4), respectively.

Note that, the Wright function $W_{\lambda,\mu}(z)$ defined by (1) does not belong to the class T. Thus, it is natural to consider the following two kinds of normalization of the Wright function:

$$\begin{aligned} W_1(\lambda,\mu;z) &= 2z - \Gamma(\mu) z W_{\lambda,\mu}(z) = 2z - \sum_{n=0}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda n+\mu)} \frac{z^{n+1}}{n!}, \\ z &\in U, \ \lambda > -1, \ \mu > 0 \end{aligned}$$

and

$$\begin{aligned} W_2(\lambda,\mu;z) &= 2z - \Gamma(\lambda+\mu)\left[W_{\lambda,\mu}(z) - \frac{1}{\Gamma(\mu)} \right] \\ &= 2z - \sum_{n=0}^{\infty} \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda n+\lambda+\mu)} \frac{z^{n+1}}{(n+1)!}, \\ z &\in U, \ \lambda > -1, \ \lambda+\mu > 0. \end{aligned}$$

From this, we can easily write:

$$W_1(\lambda,\mu;z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} \frac{z^n}{(n-1)!}, \ z \in U, \ \lambda > -1, \ \mu > 0, \tag{7}$$

$$W_2(\lambda,\mu;z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \frac{z^n}{n!}, \ z \in U, \ \lambda > -1, \ \lambda+\mu > 0. \tag{8}$$

Furthermore, we observe that the normalized Wright functions $W_1(\lambda,\mu;z)$ and $W_2(\lambda,\mu;z)$ are satisfying the following relations:

$$\lambda z(W_1(\lambda,\mu;z))' = (\mu-1)W_1(\lambda,\mu-1;z) + (\lambda-\mu+1)W_1(\lambda,\mu;z), \tag{9}$$

$$\lambda z(W_2(\lambda,\mu;z))' = (\lambda+\mu-1)W_2(\lambda,\mu-1;z) + (1-\mu)W_2(\lambda,\mu;z), \tag{10}$$

$$z(W_2(\lambda,\mu;z))' = W_1(\lambda,\lambda+\mu;z). \tag{11}$$

The main aim of the present paper is to derive several sufficient conditions for the normalized Wright functions $W_1(\lambda,\mu;z)$ and $W_2(\lambda,\mu;z)$, and for the integral transforms involving this normalized Wright functions to be in the class $T(\alpha,\beta)$ ($\alpha \in [0,1)$, $\beta \in [0,1)$).

In our present investigation, we will need of Lemma 2.1 below.

Lemma 2.1. *(see* [2, p.10, Theorem 1]*) Let the function $f \in T$ be defined by (2). Then, the function $f(z)$ is in the class $T(\alpha,\beta)$ ($\alpha \in [0,1)$, $\beta \in [0,1)$) if and only if*

$$\sum_{n=2}^{\infty} (n-\alpha)\left[\beta(n-1)+1\right] a_n \leq 1-\alpha. \tag{12}$$

3. Sufficient conditions for the normalized Wright functions

In this section, we will give some sufficient conditions for the normalized Wright functions $W_1(\lambda,\mu;z)$ and $W_2(\lambda,\mu;z)$, defined by (7) and (8) to be in the class $T(\alpha,\beta)$.

Theorem 3.1. *Let $\lambda \geq 1$, $\mu \geq \mu_0 = 0.462$ and the following condition is satisfied:*

$$(1-\alpha)(2\mu+1)(\mu+1) - \left\{(1-\alpha)(\mu+1)^2 + [1+(2-\alpha)\beta]\,(\mu+1)+\beta)\right\} e^{\frac{1}{\mu+1}} \geq 0.$$

Then, the normalized Wright function $W_1(\lambda,\mu;z)$ belongs to the class $T(\alpha,\beta)$ $(\alpha \in [0,1)\,,\ \beta \in [0,1))$.

Proof. Since

$$W_1(\lambda,\mu;z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{\Gamma((n-1)\lambda+\mu)} \frac{z^n}{(n-1)!}$$

by virtue of Lemma 2.1, it suffices to show that

$$\sum_{n=2}^{\infty} (n-\alpha)(\beta(n-1)+1)\frac{\Gamma(\mu)}{\Gamma((n-1)\lambda+\mu)} \frac{1}{(n-1)!} \leq 1-\alpha. \tag{13}$$

Let

$$L_1(\lambda,\mu;\alpha,\beta) = \sum_{n=2}^{\infty} (n-\alpha)(\beta(n-1)+1)\frac{\Gamma(\mu)}{\Gamma((n-1)\lambda+\mu)} \frac{1}{(n-1)!}.$$

We can easily write:

$$(n-\alpha)(\beta(n-1)+1) = \beta(n-2)(n-1) + (1+(2-\alpha)\beta)(n-1) + (1-\alpha).$$

In that case, by simple computation, we obtain

$$\begin{aligned} L_1(\lambda,\mu;\alpha,\beta) \quad = \quad & \sum_{n=3}^{\infty} \frac{\beta}{(n-3)!} \frac{\Gamma(\mu)}{\Gamma((n-1)\lambda+\mu)} \\ & + \sum_{n=2}^{\infty} \frac{1+(2-\alpha)\beta}{(n-2)!} \frac{\Gamma(\mu)}{\Gamma((n-1)\lambda+\mu)} \\ & + \sum_{n=2}^{\infty} \frac{1-\alpha}{(n-1)!} \frac{\Gamma(\mu)}{\Gamma((n-1)\lambda+\mu)}. \end{aligned}$$

Under the hypothesis of the theorem, for every $n \in \mathbb{N}_2 := \mathbb{N}\backslash\{1\} = \{2,3,...\}$ the inequality $\Gamma(n-1+\mu) \leq \Gamma((n-1)\lambda+\mu)$ holds true. Therefore, since $\Gamma(n-1+\mu) = \Gamma(\mu)\,(\mu)_{n-1}$, we have

$$\frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} \leq \frac{1}{(\mu)_{n-1}},\ n \in \mathbb{N}_2. \tag{14}$$

Here, $(\mu)_n = \frac{\Gamma(n+\mu)}{\Gamma(\mu)} = \mu(\mu+1)\cdots(\mu+n-1)$, $(\mu)_0 = 1$ is Pochhammer (or Appell) symbol, defined in terms of Euler gamma function.

Using (14), we have

$$L_1(\lambda,\mu;\alpha,\beta) \leq \sum_{n=3}^{\infty} \frac{\beta}{(n-3)!(\mu)_{n-1}} + \sum_{n=2}^{\infty} \frac{1+(2-\alpha)\beta}{(n-2)!(\mu)_{n-1}} + \sum_{n=2}^{\infty} \frac{1-\alpha}{(n-1)!(\mu)_{n-1}}.$$

Also, the inequality $(\mu)_{n-1} = \mu(\mu+1)\cdots(\mu+n-2) \geq \mu(\mu+1)^{n-2},\ n \in \mathbb{N}_2$ is clear, which is equivalent to

$$\frac{1}{(\mu)_{n-1}} \leq \frac{1}{\mu(\mu+1)^{n-2}},\ n \in \mathbb{N}_2. \tag{15}$$

Using (15), we obtain

$$\begin{aligned} L_1(\lambda,\mu;\alpha,\beta) &\le \sum_{n=3}^{\infty}\frac{\beta}{(n-3)!}\frac{1}{\mu(\mu+1)^{n-2}}+\sum_{n=2}^{\infty}\frac{1+(2-\alpha)\beta}{(n-2)!}\frac{1}{\mu(\mu+1)^{n-2}} \\ &\quad +\sum_{n=2}^{\infty}\frac{1-\alpha}{(n-1)!}\frac{1}{\mu(\mu+1)^{n-2}} \\ &= \left\{\frac{\beta}{\mu(\mu+1)}+\frac{1+(2-\alpha)\beta}{\mu}+\frac{(1-\alpha)(\mu+1)}{\mu}\right\}e^{\frac{1}{\mu+1}} \\ &\quad -\frac{(1-\alpha)(\mu+1)}{\mu}. \end{aligned}$$

In this case, (13) holds true if the following condition is satisfied:

$$\left\{\frac{\beta}{\mu(\mu+1)}+\frac{1+(2-\alpha)\beta}{\mu}+\frac{(1-\alpha)(\mu+1)}{\mu}\right\}e^{\frac{1}{\mu+1}}-\frac{(1-\alpha)(\mu+1)}{\mu}\le 1-\alpha,$$

which follows that

$$(1-\alpha)(2\mu+1)(\mu+1)-\left\{(1-\alpha)(\mu+1)^2+[1+(2-\alpha)\beta](\mu+1)+\beta\right\}e^{\frac{1}{\mu+1}}\ge 0.$$

Thus, the proof of Theorem 3.1 is completed. □

By setting $\beta = 0$ in Theorem 3.1 and using the first relationship in (6), we arrive at the following corollary.

Corollary 3.1. *The normalized Wright function $W_1(\lambda,\mu;z)$ belongs to the class $TS^*(\alpha)$ $(\alpha\in[0,1))$ if $\lambda\ge 1$, $\mu\ge\mu_0 = 0.462$ and the following condition is satisfied:*

$$(1-\alpha)(2\mu+1)-[(1-\alpha)(\mu+1)+1]\,e^{\frac{1}{\mu+1}}\ge 0.$$

By taking $\alpha = 0$ in Corollary 3.1, we obtain the following corollary.

Corollary 3.2. *The normalized Wright function $W_1(\lambda,\mu;z)$ belongs to the class TS^* if $\lambda\ge 1$ and $\mu\ge x_0$. Here, $x_0 = 2.4898$ is the numerical root of the equation*

$$2x+1-(x+2)e^{\frac{1}{x+1}}=0.$$

Proof. Let $\phi(x) = 2x+1-(2+x)e^{1/(x+1)}$, $x > 0$. By simple computation, we get

$$\phi'(x) = 2-\frac{x^2+x-1}{(x+1)^2}e^{\frac{1}{x+1}}.$$

As it is seen from the graphic of this function $\phi'(x) > 0$ (see Figure 1a).

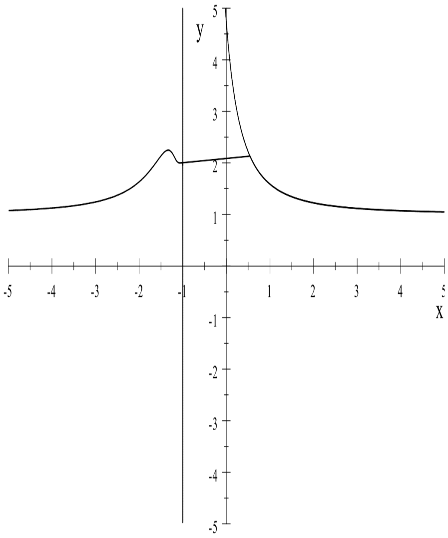

Figure 1a. Graphic of $y = \phi'(x) = 2 - \frac{x^2+x-1}{(x+1)^2} e^{\frac{1}{x+1}}$.

Thus, the function $\phi(x)$ is an increasing function.

Also, from the graphic of the function $\phi(x)$ or from the computational solution of the equation

$$2x + 1 - (x+2)e^{\frac{1}{x+1}} = 0,$$

we see that $x_0 = 2.4898$ is a numerical root of this equation (see Figure 1b and Equation 1).

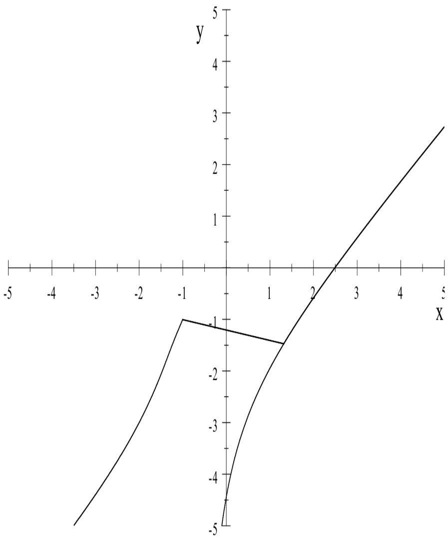

Figure 1b. Graphic of $y = \phi(x) = 2x + 1 - (x+2)e^{\frac{1}{x+1}}$.

Equation 1. $2x + 1 - (x+2)e^{\frac{1}{x+1}} = 0$. Computational numerical solution is: $x_0 = 2.4898$.

Therefore, $2\mu + 1 - (\mu + 2)e^{1/(\mu+1)} \geq 0$ for every $\mu \geq x_0$.

Thus, the proof of Corollary 3.2 is completed. □

By setting $\beta = 1$ in Theorem 3.1 and using the second relationship in (6), we arrive at the following corollary.

Corollary 3.3. *The normalized Wright function* $W_1(\lambda, \mu; z)$ *belongs to the class* $TC(\alpha)$ $(\alpha \in [0,1))$ *if* $\lambda \geq 1$, $\mu \geq \mu_0 = 0.462$ *and the following condition is satisfied:*

$$(1-\alpha)(\mu+1)(2\mu+1) - \left[(1-\alpha)(\mu+1)^2 + (3-\alpha)(\mu+1) + 1\right] e^{\frac{1}{\mu+1}} \geq 0.$$

By taking $\alpha = 0$ in Corollary 3.3, we obtain the following corollary.

Corollary 3.4. *The normalized Wright function* $W_1(\lambda, \mu; z)$ *belongs to the class* TC *if* $\lambda \geq 1$ *and* $\mu \geq x_1$. *Here,* $x_1 = 4.8523$ *is the numerical root of the equation*

$$2x^2 + 3x + 1 - (x^2 + 5x + 5)e^{\frac{1}{x+1}} = 0.$$

Proof. Let $\psi(x) = 2x^2 + 3x + 1 - (x^2 + 5x + 5)e^{\frac{1}{x+1}},\ x > 0$. By simple computation, we get

$$\psi'(x) = 4x + 3 - \frac{x(2x^2 + 8x + 7)}{(x+1)^2} e^{\frac{1}{x+1}}.$$

From the graphic of this function, we exact can see that $\psi'(x) > 0$ for each $x > 1.25$ (see Figure 2a).

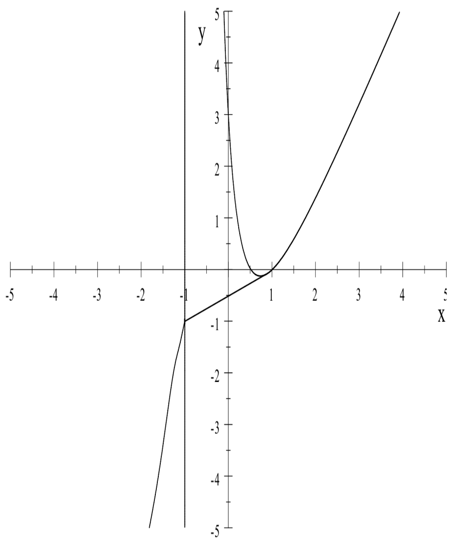

Figure 2a. Graphic of $y = \psi'(x) = 4x + 3 - \frac{x(2x^2+8x+7)}{(x+1)^2} e^{\frac{1}{x+1}}$.

Hence, the function $\psi(x)$ is an increasing function for $x > 1.25$.

Also, as it is seen from the graphic of the function $\psi(x)$ or from the computational solution of the equation

$$2x^2 + 3x + 1 - (x^2 + 5x + 5)e^{\frac{1}{x+1}} = 0$$

$x_1 = 4.8523$ is a numerical root of this equation (see Figure 2b and Equation 2).

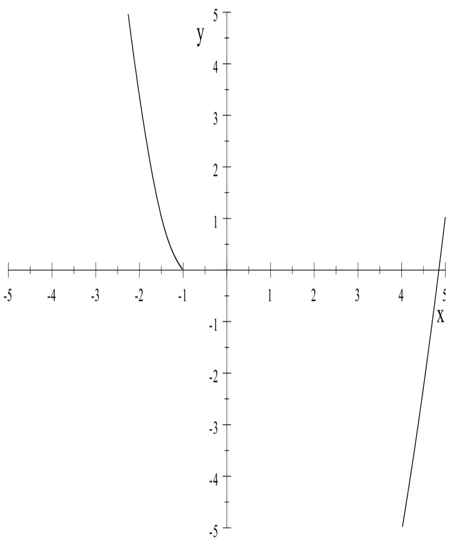

Figure 2b. Graphic of $y = \psi(x) = 2x^2 + 3x + 1 - (x^2 + 5x + 5)e^{\frac{1}{x+1}}$.

Equation 2. $2x^2 + 3x + 1 - (x^2 + 5x + 5)e^{\frac{1}{x+1}} = 0$. Computational numerical solution is:

$$x_1 = 4.8523.$$

Therefore, $2\mu^2 + 3\mu + 1 - (\mu^2 + 5\mu + 5)e^{\frac{1}{\mu+1}} \geq 0$ for every $\mu \geq x_1$.

Thus, the proof of Corollary 3.4 is completed. □

Theorem 3.2. *Let $\lambda \geq 1$, $\mu > 0$ and the following condition is satisfied:*

$$(1-\alpha)(\lambda+\mu) + (\lambda+\mu+1)\left[(1-(1-\beta)\alpha)(\lambda+\mu+2) + (1-\alpha\beta)\right]$$
$$-\left[(1-(1-\beta)\alpha)(\lambda+\mu+1)^2 + (1-\alpha\beta)(\lambda+\mu+1) + \beta\right] e^{\frac{1}{\lambda+\mu+1}} \geq 0.$$

Then, the normalized Wright function $W_2(\lambda, \mu; z)$ belongs to the class $T(\alpha, \beta)$ $(\alpha \in [0,1)\,,\ \beta \in [0,1))$.

Proof. Since

$$W_2(\lambda, \mu; z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \frac{z^n}{n!}$$

by virtue of Lemma 2.1, it suffices to show that

$$\sum_{n=2}^{\infty} (n-\alpha)(\beta(n-1)+1) \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \frac{1}{n!} \leq 1-\alpha. \tag{16}$$

Let

$$L_2(\lambda, \mu; \alpha, \beta) = \sum_{n=2}^{\infty} (n-\alpha)(\beta(n-1)+1) \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \frac{1}{n!}.$$

We can easily write: $(n-\alpha)(\beta(n-1)+1) = \beta n(n-1) + (1-\alpha\beta)n - (1-\beta)\alpha$. In that case, by simple computation, we have

$$\begin{aligned} L_2(\lambda, \mu; \alpha, \beta) \ &= \ \sum_{n=2}^{\infty} \frac{\beta}{(n-2)!} \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \\ &\quad + \sum_{n=2}^{\infty} \left(1 - \alpha\beta - \frac{1}{n}\right) \frac{1}{(n-1)!} \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \\ &\quad + \sum_{n=2}^{\infty} \frac{1-(1-\beta)\alpha}{n!} \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)}. \end{aligned}$$

Using (14) and (15), with $\mu \equiv \lambda + \mu$, we obtain

$$
\begin{aligned}
L_2(\lambda,\mu;\alpha,\beta) &\leq \sum_{n=2}^{\infty}\frac{\beta}{(n-2)!}\frac{1}{(\lambda+\mu)(\lambda+\mu+1)^{n-2}} \\
&+\sum_{n=2}^{\infty}\frac{1-\alpha\beta}{(n-1)!}\frac{1}{(\lambda+\mu)(\lambda+\mu+1)^{n-2}} \\
&+\sum_{n=2}^{\infty}\frac{1-(1-\beta)\alpha}{n!}\frac{1}{(\lambda+\mu)(\lambda+\mu+1)^{n-2}} \\
&= \frac{\beta}{\lambda+\mu}e^{\frac{1}{\lambda+\mu+1}}+\frac{(1-\alpha\beta)(\lambda+\mu+1)}{\lambda+\mu}(e^{\frac{1}{\lambda+\mu+1}}-1) \\
&+\frac{(1-(1-\beta)\alpha)(\lambda+\mu+1)^2}{\lambda+\mu}(e^{\frac{1}{\lambda+\mu+1}}-\frac{1}{\lambda+\mu+1}-1).
\end{aligned}
$$

Thus, (16) holds true if the following condition is satisfied:

$$
\left[\frac{(1-(1-\beta)\alpha)(\lambda+\mu+1)^2}{\lambda+\mu}+\frac{(1-\alpha\beta)(\lambda+\mu+1)}{\lambda+\mu}+\frac{\beta}{\lambda+\mu}\right]e^{\frac{1}{\lambda+\mu+1}}
$$
$$
-\frac{\lambda+\mu+1}{\lambda+\mu}\left[(1-(1-\beta)\alpha)(\lambda+\mu+2)+(1-\alpha\beta)\right]\leq 1-\alpha.
$$

This evidently completes the proof of Theorem 3.2. $\square$

By setting $\beta = 0$ in Theorem 3.2 and using the first relationship in (6), we arrive at the following corollary.

Corollary 3.5. *The normalized Wright function $W_2(\lambda,\mu;z)$ belongs to the class $TS^*(\alpha)$ $(\alpha\in[0,1))$ if $\lambda\geq 1$, $\mu>0$ and the following condition is satisfied:*

$$
(1-\alpha)\left[(\lambda+\mu+1)(\lambda+\mu+2)+\lambda+\mu\right]+\lambda+\mu+1
$$

$$
-\left[(1-\alpha)(\lambda+\mu+1)+1\right](\lambda+\mu+1)e^{\frac{1}{\mu+1}}\geq 0.
$$

By taking $\alpha = 0$ in Corollary 3.5, we obtain the following corollary.

Corollary 3.6. *The normalized Wright function $W_2(\lambda,\mu;z)$ belongs to the class TS^* if $\lambda\geq 1$ and $\lambda+\mu\geq x_2$. Here, $x_2 = 1.7703$ is the numerical root of the equation*

$$
x^2+5x+3-(x^2+3x+2)e^{\frac{1}{x+1}}=0.
$$

Proof. Let $h(x)=x^2+5x+3-(x^2+3x+2)e^{\frac{1}{x+1}},\ x>0$. By simple computation, we get

$$
h'(x)=2x+5-\frac{2x^2+4x+1}{x+1}e^{\frac{1}{x+1}}.
$$

As it is seen from the graphic of this function $h'(x)>0$ (see Figure 3a).

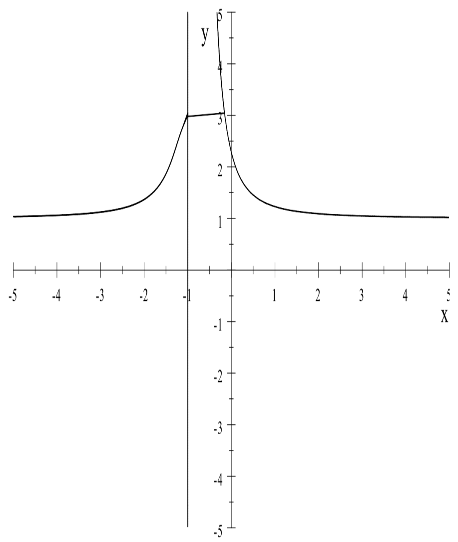

Figure 3a. Graphic of $y = h'(x) = 2x + 5 - \frac{2x^2+4x+1}{x+1} e^{\frac{1}{x+1}}$.

Thus, the function $h(x)$ is an increasing function.

Also, from the graphic of the function $h(x)$ or from the computational solution of the equation

$$x^2 + 5x + 3 - (x^2 + 3x + 2)e^{\frac{1}{x+1}} = 0,$$

we see that $x_2 = 1.7703$ is a numerical root of this equation (see Figure 3b and Equation 3).

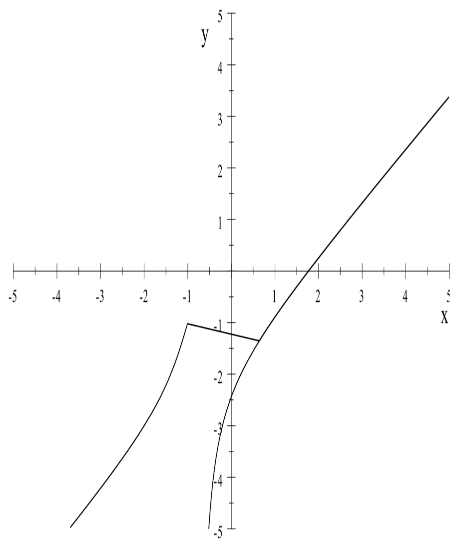

Figure 3b. Graphic of $y = h(x) = x^2 + 5x + 3 - (x^2 + 3x + 2)e^{\frac{1}{x+1}}$.

Equation 3. $x^2 + 5x + 3 - (x^2 + 3x + 2)e^{\frac{1}{x+1}} = 0$. Computational numerical solution is: $x_2 = 1.7703$.

Therefore,

$$2(\lambda + \mu) + (\lambda + \mu + 1)(\lambda + \mu + 2) + 1 - (\lambda + \mu + 1)(\lambda + \mu + 2)e^{\frac{1}{\lambda+\mu+1}} \geq 0$$

for every $\lambda + \mu \geq x_2$.

Thus, the proof of Corollary 3.6 is completed. $\square$

By setting $\beta = 1$ in Theorem 3.2 and using the second relationship in (6), we arrive at the following corollary.

Corollary 3.7. *The normalized Wright function* $W_2(\lambda, \mu; z)$ *belongs to the class* $TC(\alpha)$ $(\alpha \in [0, 1))$ *if* $\lambda \geq 1$, $\mu > 0$ *and the following condition is satisfied:*

$$(1-\alpha)\,[2(\lambda+\mu)+1] + (\lambda+\mu+1)(\lambda+\mu+2)$$

$$-\left[(1-\alpha)(\lambda+\mu+1) + (\lambda+\mu+1)^2 + 1\right] e^{\frac{1}{\lambda+\mu+1}} \geq 0.$$

By taking $\alpha = 0$ in Corollary 3.7, we obtain the following corollary.

Corollary 3.8. *The normalized Wright function* $W_2(\lambda, \mu; z)$ *belongs to the class* TC *if* $\lambda \geq 1$ *and* $\lambda + \mu \geq x_3$. *Here,* $x_3 = 2.9689$ *is the numerical root of the equation*

$$x^2 + 5x + 3 - (x^2 + 3x + 3)e^{\frac{1}{x+1}} = 0. \tag{17}$$

Proof. Let $\omega(x) = x^2 + 5x + 3 - (x^2 + 3x + 3)e^{\frac{1}{x+1}}$, $x > 0$. By simple computation, we get

$$\omega'(x) = 2x + 5 - \frac{x(2x^2 + 6x + 5)}{(x+1)^2} e^{\frac{1}{x+1}}.$$

From the graphic of this function, we see that $\omega'(x) > 0$ (see Figure 4a).

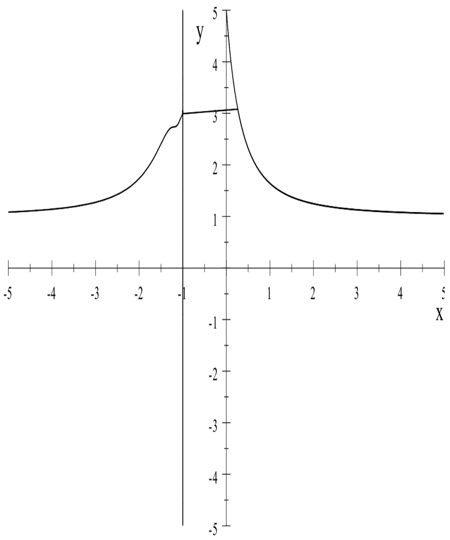

Figure 4a. Graphic of $y = \omega'(x) = 2x + 5 - \frac{x(2x^2+6x+5)}{(x+1)^2} e^{\frac{1}{x+1}}$.

Hence, the function $\omega(x)$ is an increasing function.

Also, as it is seen from the graphic of the function $\omega(x)$ or from the computational solution of the equation

$$x^2 + 5x + 3 - (x^2 + 3x + 3)e^{\frac{1}{x+1}} = 0$$

$x_3 = 2.9689$ is a numerical root of this equation (see Figure 4b and Equation 4).

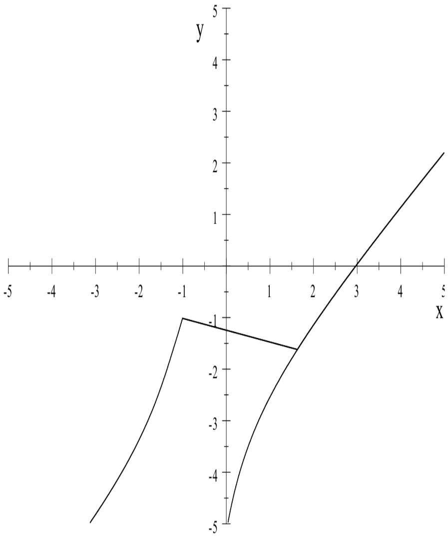

Figure 4b. Graphic of $y = \omega(x) = x^2 + 5x + 3 - (x^2 + 3x + 3)e^{\frac{1}{x+1}}$.

Equation 4. $x^2 + 5x + 3 - (x^2 + 3x + 3)e^{\frac{1}{x+1}} = 0$. Computational numerical solution is: $x_3 = 2.9686$.

Therefore,

$$(\lambda+\mu)^2 + 5(\lambda+\mu) + 3 - \left[(\lambda+\mu)^2 + 3(\lambda+\mu) + 3\right] e^{\frac{1}{\lambda+\mu+1}} \geq 0$$

for every $\lambda + \mu \geq x_3$.

Thus, the proof of Corollary 3.8 is completed. □

4. Sufficient conditions for the integrals involving normalized Wright functions

In this section, some sufficient conditions for the integrals involving the normalized Wright functions $W_1(\lambda, \mu; z)$ and $W_2(\lambda, \mu; z)$ are given.

Let

$$G_1(\lambda, \mu; z) = \int_0^z \frac{W_1(\lambda, \mu; t)}{t} dt \text{ and } G_2(\lambda, \mu; z) = \int_0^z \frac{W_2(\lambda, \mu; t)}{t} dt, \ z \in U, \tag{18}$$

where $W_1(\lambda, \mu; z)$ and $W_2(\lambda, \mu; z)$ are functions, defined by (7) and (8), respectively. Note that $G_1, G_2 \in T$.

In the next theorems, we give sufficient conditions so that $G_1(\lambda, \mu; z)$ and $G_2(\lambda, \mu; z)$ are in the class $T(\alpha, \beta)$.

Theorem 4.1. *Let $\lambda \geq 1$, $\mu \geq \mu_0 = 0.462$ and the following condition is satisfied:*

$$(1-\alpha)\mu + \left[(1-(1-\beta)\alpha)(\mu+2) + 1 - \alpha\beta\right](\mu+1)$$
$$- \left[(1-(1-\beta)\alpha)(\mu+1)^2 + (1-\alpha\beta)(\mu+1) + \beta\right] e^{\frac{1}{\mu+1}} \geq 0. \tag{19}$$

Then, the function $G_1(\lambda, \mu; z)$ belongs to the class $T(\alpha, \beta)$ $(\alpha \in [0,1), \beta \in [0,1))$.

Proof. Our proof of Theorem 4.1 is similar of that of Theorem 3.2. Indeed, from the definition of function $G_1(\lambda, \mu; z)$, we can easily see that

$$G_1(\lambda, \mu; z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} \frac{z^n}{n!} = W_2(\lambda, \mu - \lambda; z).$$

Therefore, the details of the proof of Theorem 4.1 may be omitted. □

By setting $\beta = 0$ in Theorem 4.1 and using the first relationship in (6), we arrive at the following corollary.

Corollary 4.1. *The function $G_1(\lambda, \mu; z)$ belongs to the class $TS^*(\alpha)$ ($\alpha \in [0,1)$) if $\lambda \geq 1$, $\mu \geq \mu_0 = 0.462$ and the following condition is satisfied:*

$$(1-\alpha)\left[(\mu+1)^2 + 2\mu + 1\right] + \mu + 1 - \left[(1-\alpha)(\mu+1) + 1\right](\mu+1)e^{\frac{1}{\mu+1}} \geq 0.$$

By taking $\alpha = 0$ in Corollary 4.1, we obtain the following corollary.

Corollary 4.2. *The function $G_1(\lambda, \mu; z)$ belongs to the class TS^* if $\lambda \geq 1$ and $\mu \geq x_2$. Here, $x_2 = 1.7703$ is the numerical root of the equation*

$$x^2 + 5x + 3 - (x^2 + 3x + 2)e^{\frac{1}{x+1}} = 0.$$

Proof. The proof of Corollary 4.2 is very similar of the proof of Corollary 3.6. Therefore, the details of the proof of Corollary 4.1 may be omitted. □

By setting $\beta = 1$ in Theorem 4.1, and using the second relationship in (6), we arrive at the following corollary.

Corollary 4.3. *The function $G_1(\lambda, \mu; z)$ belongs to the class $TC(\alpha)$ ($\alpha \in [0,1)$) if $\lambda \geq 1$, $\mu \geq \mu_0 = 0.462$ and the following condition is satisfied:*

$$(1-\alpha)(2\mu+1) + (\mu+1)(\mu+2) - \left[(\mu+2-\alpha)(\mu+1) + 1\right] e^{\frac{1}{\mu+1}} \geq 0.$$

By taking $\alpha = 0$ in Corollary 4.3, we obtain the following corollary.

Corollary 4.4. *Let $\lambda \geq 1$ and $\mu \geq x_3$, where $x_3 = 2.9689$ is the numerical root of the equation (17), then $G_1 \in TC$.*

Proof. The proof of Corollary 4.4 is the same of the proof of Corollary 3.8. □

Theorem 4.2. *Let $\lambda \geq 1$, $\mu > 0$ and the following condition is satisfied:*

$$\begin{aligned} &(1-\alpha)(\lambda+\mu) + (2-(1+\beta)\alpha)(\lambda+\mu+1)(\lambda+\mu+2) + (\lambda+\mu+1)\beta \\ &- \left[(2-(1+\beta)\alpha)(\lambda+\mu+1) + \beta\right](\lambda+\mu+1)e^{\frac{1}{\lambda+\mu+1}} \geq 0. \end{aligned} \tag{20}$$

Then, the function $G_2(\lambda, \mu; z)$ belongs to the class $T(\alpha, \beta)$ ($\alpha \in [0,1)$, $\beta \in [0,1)$).

Proof. Since

$$G_2(\lambda, \mu; z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \frac{z^n}{n \cdot n!}$$

by virtue of Lemma 2.1, it suffices to show that

$$\sum_{n=2}^{\infty} (n-\alpha)(\beta(n-1)+1) \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \frac{1}{n \cdot n!} \leq 1-\alpha. \tag{21}$$

Let

$$L_3(\lambda, \mu; \alpha, \beta) = \sum_{n=2}^{\infty} (n-\alpha)(\beta(n-1)+1) \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \frac{1}{n \cdot n!}.$$

We can easily write: $(n-\alpha)(\beta(n-1)+1) = n^2\beta + (1-\alpha\beta)n - n\beta - (1-\beta)\alpha$. Hence, by simple computation, we get

$$
\begin{aligned}
L_3(\lambda,\mu;\alpha,\beta) &= \sum_{n=2}^{\infty}(1-\frac{1}{n})\frac{\beta}{(n-1)!}\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \\
&\quad+\sum_{n=2}^{\infty}(1-\frac{1}{n})\frac{1-\alpha\beta}{n!}\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \\
&\quad+\sum_{n=2}^{\infty}\frac{1-\alpha}{n\cdot n!}\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)}.
\end{aligned}
$$

By using (14) and (15), with $\mu \equiv \lambda+\mu$, we get

$$
\begin{aligned}
L_3(\lambda,\mu;\alpha,\beta) &\leq \sum_{n=2}^{\infty}\frac{\beta}{(n-1)!}\frac{1}{(\lambda+\mu)(\lambda+\mu+1)^{n-2}} \\
&\quad+\sum_{n=2}^{\infty}\frac{1-\alpha\beta}{n!}\frac{1}{(\lambda+\mu)(\lambda+\mu+1)^{n-2}} \\
&\quad+\sum_{n=2}^{\infty}\frac{1-\alpha}{n\cdot n!}\frac{1}{(\lambda+\mu)(\lambda+\mu+1)^{n-2}} \\
&= \frac{1}{\lambda+\mu}\left\{\{[2-(1+\beta)\alpha]\,(\lambda+\mu+1)+\beta\right\}(\lambda+\mu+1)e^{\frac{1}{\lambda+\mu+1}} \\
&\quad-\{[2-(1+\beta)\alpha]\,(\lambda+\mu+2)+\beta\}\,(\lambda+\mu+1)\}\,.
\end{aligned}
$$

We easily see that (21) holds true if the following condition is satisfied:

$$
\begin{aligned}
&\frac{1}{\lambda+\mu}\left\{\{[2-(1+\beta)\alpha]\,(\lambda+\mu+1)+\beta\right\}(\lambda+\mu+1)e^{\frac{1}{\lambda+\mu+1}} \\
&-\{[2-(1+\beta)\alpha]\,(\lambda+\mu+2)+\beta\}\,(\lambda+\mu+1)\} \leq 1-\alpha,
\end{aligned}
$$

which is equivalent to (20).

Thus, the proof of Theorem 4.2 is completed. □

By setting $\beta = 0$ in Theorem 4.2 and using the first relationship in (6), we arrive at the following corollary.

Corollary 4.5. *The function $G_2(\lambda,\mu;z)$ belongs to the class $TS^*(\alpha)$ $(\alpha \in [0,1))$ if $\lambda \geq 1$, $\mu > 0$ and the following condition is satisfied:*

$$(1-\alpha)(\lambda+\mu)+(2-\alpha)(\lambda+\mu+1)(\lambda+\mu+2)-(2-\alpha)(\lambda+\mu+1)^2e^{\frac{1}{\lambda+\mu+1}} \geq 0.$$

By taking $\alpha = 0$ in Corollary 4.5, we obtain the following corollary.

Corollary 4.6. *The function $G_2(\lambda,\mu;z)$ belongs to the class TS^* if $\lambda \geq 1$ and $\lambda+\mu \geq x_4$. Here, $x_4 = 1.1728$ is the numerical root of the equation*

$$2x^2+7x+4-2(x+1)^2e^{\frac{1}{x+1}} = 0.$$

Proof. Let $\sigma(x) = 2x^2+7x+4-2(x+1)^2e^{\frac{1}{x+1}},\ x > 0$. By simple computation, we get

$$\sigma'(x) = 4x+7-2(2x+1)e^{\frac{1}{x+1}}.$$

From the graphic of this function, we see that $\sigma'(x) > 0$ (see Figure 5a).

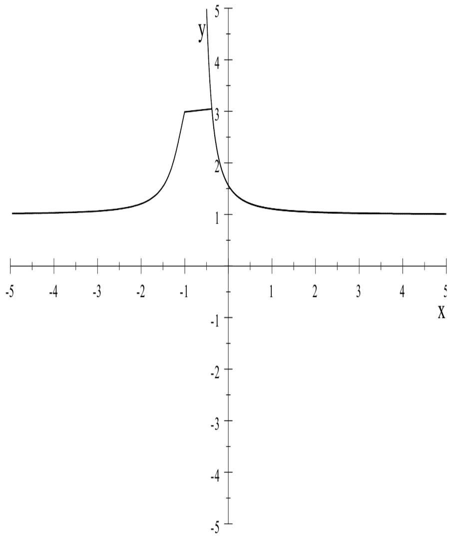

Figure 5a. Graphic of $y = \sigma'(x) = 4x + 7 - 2(2x+1)e^{\frac{1}{x+1}}$.

Thus, the function $\sigma(x)$ is an increasing function.

Also, as it is seen from the graphic of the function $\sigma(x)$ or from the computational solution of the equation

$$2x^2 + 7x + 4 - 2(x+1)^2 e^{\frac{1}{x+1}} = 0$$

$x_4 = 1.1728$ is a numerical root of this equation (see Figure 5b and Equation 5).

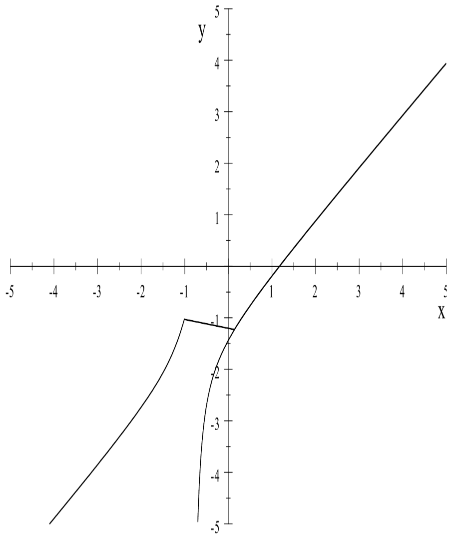

Figure 5b. Graphic of $y = \sigma(x) = 2x^2 + 7x + 4 - 2(x+1)^2 e^{\frac{1}{x+1}}$.

Equation 5. $2x^2 + 7x + 4 - 2(x+1)^2 e^{\frac{1}{x+1}} = 0$. Computational numerical solution is: $x_4 = 1.1728$.

Therefore,

$$(\lambda + \mu) - 2(\lambda + \mu + 1)^2 e^{\frac{1}{\lambda+\mu+1}} + 2(\lambda + \mu + 1)(\lambda + \mu + 2) \geq 0$$

for every $\lambda + \mu \geq x_4$.

Thus, the proof of Corollary 4.6 is completed. $\square$

By setting $\beta = 1$ in Theorem 4.2, and using the second relationship in (6), we arrive at the following corollary.

Corollary 4.7. *The function $G_2(\lambda, \mu; z)$ belongs to the class $TC(\alpha)$ $(\alpha \in [0, 1))$ if $\lambda \geq 1$, $\mu > 0$ and the following condition is satisfied:*

$$(1-\alpha)\left[2(\lambda+\mu+1)(\lambda+\mu+2)+\lambda+\mu\right]+\lambda+\mu+1$$
$$-(\lambda+\mu+1)\left[2(1-\alpha)(\lambda+\mu+1)+1\right]e^{\frac{1}{\lambda+\mu+1}} \geq 0.$$

By taking $\alpha = 0$, in Corollary 4.7, we obtain the following corollary.

Corollary 4.8. *The function $G_2(\lambda, \mu; z)$ belongs to the class TC if $\lambda \geq 1$ and $\lambda+\mu \geq x_5$. Here, $x_5 = 2.2791$ is the numerical root of the equation*

$$2x^2+8x+5-(2x^2+5x+3)e^{\frac{1}{x+1}} = 0.$$

Proof. The proof of Corollary 4.8 is very similar of the proof of the above corollaries. Therefore, the proof of this corollary may be omitted. □

5. Concluding remarks and observations

In our presented investigation, we have systematically studied two new kinds of normalization of the Wright function and integrals involving these functions. Our main results obtained in Theorems 3.1, 3.2, 4.1 and 4.2 are new and their usefulness is shown by deducing several interesting corollaries and examples. We have also considered relevant connections of our results with various earlier related results.

Disclosure statement. The author declares that there is not conflict of interests regarding the publication of this paper.

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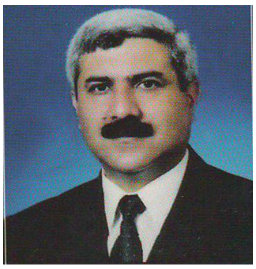

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