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# DESIGN AND ANALYSIS OF A CLASS OF WEIGHTED-NEWTON METHODS WITH FROZEN DERIVATIVE

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Abstract. We present a generalized $k$-step scheme of weighted-Newton methods with increasing convergence order $2k+2$ for nonlinear equations. The novelty of the scheme is that in each step the order of convergence is increased by an amount of two at the cost of only one additional function evaluation. The algorithm requires only single evaluation of Fréchet derivative which points to the name 'methods with frozen derivative'. Local convergence including radius of convergence, error bounds and estimates on the uniqueness of the solution is presented. To maximize the computational efficiency, the optimal number of steps is computed. Theoretical results regarding convergence and computational efficiency are verified through numerical experimentation.

Keywords: nonlinear equations, weighted-Newton methods, fast algorithms, convergence, computational efficiency.

AMS Subject Classification: 49M15, 41A25, 65B99, 65J15.

## 1. Introduction

In this paper, we consider the problem of approximating a solution $x^*$ of the equation $F(x) = 0$; where $F : \Omega \subseteq B_1 \to B_2$, $B_1$ and $B_2$ are Banach spaces and $\Omega$ is a nonempty open convex subset of $B_1$. Many problems in computational sciences can be written in the form $F(x) = 0$ using Mathematical Modelling (for example [1, 2]). The solution of these equations can be found in closed form only in special cases. That explains why most methods for solving such equations are usually iterative. The important part in the development of an iterative method is to study its convergence analysis. This is usually divided into two categories viz. semilocal and local convergence. The semilocal convergence is based on the information around an initial point and gives criteria that ensures the convergence of iteration procedures. Local convergence is based on the information of convergence domain. In general the convergence domain is small. Therefore, it is important to enlarge the convergence domain without additional hypothesis. Another important problem is to find more precise error estimates on $\|x_{n+1} - x_n\|$ or $\|x_n - x^*\|$. There exist many studies which deal with the local and semilocal convergence analysis of iterative methods such as [1, 3, 6, 13, 16, 18].

The most widely used iterative method for solving $F(x) = 0$ is the quadratically convergent Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad n = 0, 1, 2, \dots, \tag{1}$$

where $F'(x)^{-1}$ is the inverse of first Fréchet derivative $F'(x)$ of the function $F(x)$. In order to accelerate the convergence, researchers have also obtained some modified Newton's or Newton-like methods, see [5, 4, 7, 8, 10, 21, 19, 20] and references there in. In particular, recently Sharma

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and Arora [19] have developed weighted-Newton methods of fourth and sixth order convergence, which are given by

$$x_{n+1} = y_n - (3I - 2F'(x_n)^{-1}[y_n, x_n;\, F])F'(x_n)^{-1}F(y_n) \tag{2}$$

and

$$\begin{aligned} z_n &= y_n - (3I - 2F'(x_n)^{-1}[y_n, x_n;\, F])F'(x_n)^{-1}F(y_n), \\ x_{n+1} &= z_n - (3I - 2F'(x_n)^{-1}[y_n, x_n;\, F])F'(x_n)^{-1}F(z_n), \end{aligned} \tag{3}$$

where $y_n = x_n - F'(x_n)^{-1}F(x_n)$ is Newton iteration and $[y, x;\, F]$ is first order divided difference of $F$. Per iteration the fourth order method (2) utilizes two functions, one derivative, one divided difference and one inverse where as the sixth order method (3) requires one additional function evaluation to that of the evaluations of fourth order method. The notable point for these methods is that they use not only single derivative but also single inverse operator.

In this paper, based on the idea of fourth order scheme (2) we present a generalized $k$-step scheme with increasing convergence order $2k + 2$ ($k \in \mathbb{N}$). The methods (2) and (3) are special cases corresponding to $k = 1$ and $k = 2$. The novel feature of the scheme is that in each step the order of convergence is increased by an amount two at the cost of only one additional function evaluation. The evaluations of derivative $F'(x_n)$ and its inverse remain the same through out which also points to the name 'methods with frozen derivative'.

The rest of the paper is structured as follows. In section 2, we present the generalized method with its order of convergence and local convergence. In section 3, the optimal number of steps in order to maximize the computational efficiency index is computed. Theoretical results concerning local convergence are verified through numerical examples in section 4. The applicability of different members of the family to solve numerical problems is tested in section 5.

## 2. The method and its convergence

Consider a general scheme of weighted-Newton methods

$$\begin{aligned} &w_{n,1} = w_{n,0} - \psi(x_n, w_{n,0})F'(x_n)^{-1}F(w_{n,0}), \\ &w_{n,2} = w_{n,1} - \psi(x_n, w_{n,0})F'(x_n)^{-1}F(w_{n,1}), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &w_{n,k-1} = w_{n,k-2} - \psi(x_n, w_{n,0})F'(x_n)^{-1}F(w_{n,k-2}), \\ &w_{n,k} = x_{n+1} = w_{n,k-1} - \psi(x_n, w_{n,0})F'(x_n)^{-1}F(w_{n,k-1}), \end{aligned} \tag{4}$$

where $k \in \mathbb{N}$, $w_{n,0} = \; x_n - F'(x_n)^{-1}F(x_n)$ and $\psi(x_n, w_{n,0}) = 3I - 2F'(x_n)^{-1}[w_{n,0}, x_n;\, F]$. This is a $k$-step scheme which includes the methods (2) and (3) as the special cases for $k = 1$ and $k = 2$, respectively. It is clear that the scheme requires the information of $k + 1$ functions, one derivative, one divided difference and one inverse operator per iteration.

Next, the order of convergence and local convergence of (4) are presented.

2.1. **Order of convergence.** We find the convergence order of method (4), when $F : \Omega \subset \mathbb{R}^m \to \mathbb{R}^m$. In order to examine convergence order of the scheme (4), the definition of divided difference is required. For this, recalling the following result of Taylor's expansion on vector functions ([14]):

**Lemma 2.1.** *$F:\Omega\subset\mathbb{R}^m\to\mathbb{R}^m$ be $r$-times Fréchet differentiable in a convex set $\Omega\subset\mathbb{R}^m$ then for any $x,h\in\mathbb{R}^m$, the following expression holds:*

$$F(x+h)=F(x)+F'(x)h+\frac{1}{2!}F''(x)h^2+\frac{1}{3!}F'''(x)h^3+...+\frac{1}{(r-1)!}F^{(r-1)}(x)h^{r-1}+R_r, \quad (5)$$

*where*

$$||R_r||\leqslant\frac{1}{r!}\sup_{0\leqslant t\leqslant 1}||F^{(r)}(x+th)||\,||h||^r \text{ and } h^r=(h,h,\overset{r}{\ldots},h).$$

The divided difference operator of $F$ is a mapping $[\cdot,\cdot\,;F]:\Omega\times\Omega\subset\mathbb{R}^m\times\mathbb{R}^m\longrightarrow L(\mathbb{R}^m)$ defined by ([17])

$$[x+h,x\,;F]=\int_0^1F'(x+th)\,dt,\ \forall\,x,h\in\mathbb{R}^m. \quad (6)$$

Expanding $F'(x+th)$ in Taylor series at the point $x$ and integrating, we have

$$[x+h,x\,;F]=\int_0^1F'(x+th)\,dt=F'(x)+\frac{1}{2}F''(x)h+\frac{1}{6}F'''(x)h^2+O(h^3). \quad (7)$$

where $h^i=(h,h,\overset{i}{\ldots},h)$, $h\in\mathbb{R}^m$.

Let $e_n=x_n-x^*$. Developing $F(x_n)$ in a neighborhood of $x^*$ and assuming that $\Gamma=F'(x^*)^{-1}$ exists, we have

$$F(x_n)=F'(x^*)(e_n+A_2(e_n)^2+A_3(e_n)^3+A_4(e_n)^4+O((e_n)^5)), \quad (8)$$

where $A_i=\frac{1}{i!}\Gamma F^{(i)}(x^*)\in L_i(\mathbb{R}^m,\mathbb{R}^m)$ and $(e_n)^i=(e_n,e_n,\overset{i}{\ldots},e_n)$, $e_n\in\mathbb{R}^m,\ i=2,3,\ldots$. Also,

$$F'(x_n)=F'(x^*)(I+2A_2e_n+3A_3(e_n)^2+4A_4(e_n)^3+O((e_n)^4)), \quad (9)$$

$$F''(x_n)=F'(x^*)(2A_2+6A_3e_n+12A_4(e_n)^2+O((e_n)^3)), \quad (10)$$

$$F'''(x_n)=F'(x^*)(6A_3+24A_4e_n+O((e_n)^2)). \quad (11)$$

Inversion of $F'(x_n)$ yields,

$$F'(x_n)^{-1}=(I-2A_2e_n+(4A_2^2-3A_3)(e_n)^2-(4A_4-6A_2A_3-6A_3A_2+8A_2^3)(e_n)^3 + O((e_n)^4))\Gamma. \quad (12)$$

**Theorem 2.1.** *Let $F:\Omega\subset\mathbb{R}^m\to\mathbb{R}^m$ be a sufficiently many times differentiable mapping. Then, sequence $\{x_n\}$ generated by method (4) for $x_0\in\Omega$ converges to $x^*$ with order $2k+2$ for $k\in\mathbb{N}$.*

**Proof.** Using Eqs. (9)–(11) in (7) for $x+h=w_{n,0}$, $x=x_n$ and $h=\bar{e}_n-e_n$, it follows that

$$[w_{n,0},x_n;\,F]=\ F'(x^*)\big(I+A_2(\bar{e}_n+e_n)+A_3(e_n)^2+O((e_n)^3)\big), \quad (13)$$

where $\bar{e}_n$ is the local error of Newton's method given by

$$\begin{aligned}\bar{e}_n&=w_{n,0}-x^*\\&=A_2(e_n)^2+2(A_3-A_2^2)(e_n)^3+(3A_4-4A_2A_3-3A_3A_2+4A_2^3)(e_n)^4+O((e_n)^5).\end{aligned} \quad (14)$$

From (12) and (13), we obtain

$$\begin{aligned}\psi(x_n,w_{n,0})&=3I-2F'(x_n)^{-1}[w_{n,0},x_n;\ F]\\&=I+2A_2e_n-2(3A_2^2-2A_3)(e_n)^2+O((e_n)^3).\end{aligned} \quad (15)$$

Post-multiplying (15) by $F'(x_n)^{-1}$, we obtain that

$$\psi(x_n,w_{n,0})F(x_n)^{-1}=\big(I+(A_3-6A_2^2)(e_n)^2+O((e_n)^3)\big)\Gamma. \quad (16)$$

Taylor's expansion of $F(w_{n,k-1})$ about $x^*$ yields

$$F(w_{n,k-1}) = F'(x^*)\big((w_{n,k-1} - x^*) + A_2(w_{n,k-1} - x^*)^2 + \cdots\big). \tag{17}$$

Then, we have that

$$\begin{aligned}\psi(x_n, w_{n,0})F(x_n)^{-1}F(w_{n,k-1}) &= \big(I + (A_3 - 6A_2^2)(e_n)^2 + O((e_n)^3)\big)F'(x^*)^{-1} \\ &\quad \times F'(x^*)\big((w_{n,k-1} - x^*) + A_2(w_{n,k-1} - x^*)^2 + \cdots\big) \\ &= (w_{n,k-1} - x^*) + (A_3 - 6A_2^2)(e_n)^2(w_{n,k-1} - x^*) \\ &\quad + A_2(w_{n,k-1} - x^*)^2 + \cdots .\end{aligned} \tag{18}$$

Using (18) in last step of (4), it follows that

$$w_{n,k} - x^* = \big(6A_2^2 - A_3)(e_n)^2(w_{n,k-1} - x^*) + A_2(w_{n,k-1} - x^*)^2 + \cdots . \tag{19}$$

By the proof of Theorem 1 proved in [19], we can write

$$w_{n,1} - x^* = (5A_2^3 - A_3A_2)(e_n)^4 + O((e_n)^5),$$

therefore, from (19) for $k = 2, 3$, we have

$$\begin{aligned}w_{n,2} - x^* &= (6A_2^2 - A_3)(e_n)^2(w_{n,1} - x^*) + \cdots \\ &= (6A_2^2 - A_3)(e_n)^2(5A_2^3 - A_3A_2)(e_n)^4 + O\big((e_n)^7\big) \\ &= (6A_2^2 - A_3)(5A_2^3 - A_3A_2)(e_n)^6 + O\big((e_n)^7\big)\end{aligned}$$

and

$$\begin{aligned}w_{n,3} - x^* &= (6A_2^2 - A_3)(e_n)^2(w_{n,2} - x^*) + \cdots \\ &= (6A_2^2 - A_3)(e_n)^2(6A_2^2 - A_3)(5A_2^3 - A_3A_2)(e_n)^6 + O\big((e_n)^9\big) \\ &= (6A_2^2 - A_3)^2(5A_2^3 - A_3A_2)(e_n)^8 + O\big((e_n)^9\big).\end{aligned}$$

Proceeding by induction, we have

$$e_{n+1} = w_{n,k} - x^* = (6A_2^2 - A_3)^{k-1}(5A_2^3 - A_3A_2)(e_n)^{2k+2} + O\big((e_n)^{2k+3}\big). \qquad \square \tag{20}$$

2.2. **Local convergence.** We study the convergence of generalized family of methods (4) in Banach space setting. Let $\omega_0 : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}$ be a continuous and nondecreasing function with $\omega_0(0) = 0$. Let also $r$ be such that

$$r = \sup\{t \ge 0 : w_0(t) < 1\}. \tag{21}$$

Consider, functions $v, v_0, \omega : [0, r) \to \mathbb{R}_+ \cup \{0\}$ continuous and nondecreasing with $\omega(0) = 0$. Define functions $g_0$ and $h_0$ on the interval $[0, r)$ by

$$g_0(t) = \frac{\int_0^1 \omega((1-\theta)t)d\theta}{1 - \omega_0(t)}$$

and

$$h_0(t) = g_0(t) - 1.$$

We have $h_0(0) = -1 < 0$ and $h_0(t) \to +\infty$ as $t \to r^-$. The intermediate value theorem guarantees that equation $h_0(t) = 0$ has solution in $(0, r)$. Denote by $r_0$ the smallest such solution. Define functions $\bar{g}$, $\lambda$, $\mu_i$ and $h_i$ on the interval $[0, r)$ by

$$\bar{g}(t) = 1 + \frac{2\big(\omega_0(t) + v_0[(1 + g_0(t))t]\big)}{1 - \omega_0(t)},$$

$$\lambda(t) = 1 + \frac{\bar{g}(t)\int_0^1 v(\theta g_0(t)t)d\theta}{1-\omega_0(t)},$$

$$\mu_i(t) = \lambda^i(t)g_0(t), \quad i = 1, 2, \dots, k$$

and

$$h_i(t) = \mu_i(t) - 1.$$

We have that $h_i(0) < 0$. Suppose that

$$\mu_i(t) \to +\infty \text{ or a positive number as } t \to r^-. \tag{22}$$

Denote by $r^{(i)}$ the smallest zero of function $h_i$ on the interval $(0, r)$. Define the radius of convergence $r^*$ by

$$r^* = \min\{r_0, r^{(i)}\}. \tag{23}$$

Then, we have that for each $t \in [0, r^*)$

$$0 \leq \mu_i(t) < 1, \ \ i = 1, 2, \dots, k. \tag{24}$$

Denote by $U(\nu, \varepsilon) = \{x \in B_1 : \|x - \nu\| < \varepsilon\}$ the ball with center $\nu \in B_1$ and of radius $\varepsilon > 0$. Moreover, $\bar{U}(\nu, \varepsilon)$ denotes the closure of $U(\nu, \varepsilon)$.
We shall show the local convergence analysis of method (4) in a Banach space setting under hypotheses (A):

- (a1) $F : \Omega \subseteq B_1 \to B_2$ is a continuously Fréchet-differentiable operator and $[\cdot, \cdot; F] : \Omega \times \Omega \to \mathcal{L}(B_1, B_2)$ be a divided difference operator of $F$.
- (a2) There exists $x^* \in \Omega$ such that $F(x^*) = 0$ and $F'(x^*)^{-1} \in \mathcal{L}(B_2, B_1)$.
- (a3) There exists function $\omega_0 : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}$ continuous and nondecreasing with $\omega_0(0) = 0$ such that for each $x \in \Omega$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq \omega_0(\|x - x^*\|).$$

- (a4) Let $\Omega_0 = \Omega \cap U(x^*, r)$, where $r$ was defined in (21). There exist functions $\omega : [0, r) \to \mathbb{R}_+ \cup \{0\}$, $v_0, v : [0, r) \to \mathbb{R}_+ \cup \{0\}$ continuous and nondecreasing with $\omega(0) = 0$ such that for each $x, y \in \Omega_0$

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq \omega(\|x - y\|)$$

and

$$\|F'(x^*)^{-1}(F'(x^*) - [x, y;\, F])\| \leq v_0(\|x - x^*\| + \|y - x^*\|).$$

- (a5) $\bar{U}(x^*, r^*) \subseteq \Omega$ and $\|F'(x^*)^{-1}F'(x)\| \leq v(\|x - x^*\|)$.
- (a6) $\int_0^1 w_0(\theta R)d\theta < 1$ for some $R \geq r^*$. Set $\Omega_1 = \Omega \cap \bar{U}(x^*, R)$.

**Theorem 2.2.** *Suppose that the conditions (A) hold. Then, sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r^*) - \{x^*\}$ by method (4) is well defined in $U(x^*, r^*)$, remains in $U(x^*, r^*)$ and converges to $x^*$. Moreover, the following estimates hold*

$$\|w_{n,0} - x^*\| \leq g_0(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r^*, \tag{25}$$

$$\begin{aligned}\|w_{n,i} - x^*\| &\leq \lambda^i(\|x_n - x^*\|)\|w_{n,0} - x^*\| \\ &\leq \lambda^i(\|x_n - x^*\|)g_0(\|x_n - x^*\|)\|x_n - x^*\| \\ &\leq \|x_n - x^*\|, \quad i = 1, 2, \dots, k-1\end{aligned} \tag{26}$$

*and*

$$\|x_{n+1}-x^*\| = \|w_{n,k}-x^*\| \le \lambda^k(\|x_n-x^*\|)\|w_{n,0}-x^*\|, \\ \le \mu_k(\|x_n-x^*\|)\|x_n-x^*\|, \quad (27)$$

*where the functions $\lambda$ and $\mu$ are defined previously. Furthermore, the vector $x^*$ is the only solution of equation $F(x)=0$ in $\Omega_1$.*

**Proof.** We shall show estimates (25)–(27) using mathematical induction. By hypothesis (a3) and for $x \in U(x^*, r^*)$, we have that

$$\|F'(x^*)^{-1}(F'(x)-F'(x^*))\| \le \omega_0(\|x-x^*\|) \le \omega_0(r^*) < 1. \quad (28)$$

By the Banach perturbation Lemma [1] and (28), we get that $F'(x)^{-1} \in \mathcal{L}(B_2, B_1)$ and

$$\|F'(x)^{-1}F'(x^*)\| \le \frac{1}{1-\omega_0(\|x-x^*\|)}. \quad (29)$$

In particular, (29) holds for $x = x_n$, since $x_n \in U(x^*, r^*) - \{x^*\}$. By using (4) and (a2) We can write that

$$\begin{aligned} w_{n,0}-x^* &= x_n - x^* - F'(x_n)^{-1}F(x_n) \\ &= \int_0^1 F'(x_n)^{-1}\big(F'(x^*+\theta(x_n-x^*)) - F'(x_n)\big)(x_n-x^*)d\theta. \end{aligned} \quad (30)$$

Then, using (24) (for $i=0$), the first condition in (a4), (29) (for $x=x_n$) and (30) we get in turn that

$$\begin{aligned} \|w_{n,0}-x^*\| &= \|F'(x_n)^{-1}F'(x^*)\| \left\| \int_0^1 F'(x^*)^{-1}[F'(x^*+\theta(x_n-x^*)-F'(x_n)](x_n-x^*)d\theta \right\| \\ &= \frac{\int_0^1 \omega((1-\theta)\|x_n-x^*\|)d\theta\|x_n-x^*\|}{1-\omega_0(\|x_n-x^*\|)} \\ &= g_0(\|x_n-x^*\|)\|x_n-x^*\| \le \|x_n-x^*\| < r^*, \end{aligned} \quad (31)$$

which implies (25) and $w_{n,0} \in U(x^*, r^*)$. We have $\psi(x_n, w_{n,0}) = 3I - 2F'(x_n)^{-1}[w_{n,0}, x_n; F]$ and we get that

$$\begin{aligned} \|\psi(x_n, w_{n,0})\| &= \|3I - 2F'(x_n)^{-1}[w_{n,0}, x_n; F]\| \\ &\le 1 + \|2F'(x_n)^{-1}\big(F'(x_n) - [w_{n,0}, x_n; F]\big)\| \\ &\le 1 + 2\|F'(x_n)^{-1}F'(x^*)\|\big(\|F'(x^*)^{-1}(F'(x_n)-F'(x^*))\| \\ &\quad + \|F'(x^*)^{-1}(F'(x^*) - [w_{n,0}, x_n; F])\|\big) \\ &\le 1 + \frac{2\big(\omega_0(\|x_n-x^*\|) + v_0\big(\|x_n-x^*\| + \|w_{n,0}-x^*\|\big)\big)}{1-\omega_0(\|x_n-x^*\|)} \\ &\le 1 + \frac{2\big(\omega_0(\|x_n-x^*\|) + v_0\big(\big(1+g_0(\|x_n-x^*\|)\big)\|x_n-x^*\|\big)\big)}{1-\omega_0\|x_n-x^*\|} \\ &\le \bar{g}(\|x_n-x^*\|). \end{aligned} \quad (32)$$

By using (a6), we get that

$$\begin{aligned} \|F'(x^*)^{-1}F(x_n)\| &= \left\| \int_0^1 F'(x^*)^{-1}F'(x^*+\theta(x_n-x^*))(x_n-x^*)d\theta \right\| \\ &\le \int_0^1 v(\theta\|x_n-x^*\|)d\theta\|x_n-x^*\|. \end{aligned} \quad (33)$$

By using (32) and (33), we have in turn the estimates

$$\begin{aligned}
\|w_{n,1}-x^*\| &= \|w_{n,0}-x^*-\psi(x_n,w_{n,0})F(x_n)^{-1}F(w_{n,0})\| \\
&\le \|w_{n,0}-x^*\| + \|\psi(x_n,w_{n,0})F(x_n)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(w_{n,0})\| \\
&\le \|w_{n,0}-x^*\| + \frac{\bar{g}(\|x_n-x^*\|)}{1-\omega_0(\|x_n-x^*\|)}\int_0^1 v(\theta\|w_{n,0}-x^*\|)d\theta\|w_{n,0}-x^*\| \\
&= \left(1+\frac{\bar{g}(\|x_n-x^*\|)\int_0^1 v(\theta\|w_{n,0}-x^*\|)d\theta}{1-\omega_0(\|x_n-x^*\|)}\right)\|w_{n,0}-x^*\| \\
&\le \lambda(\|x_n-x^*\|)\|w_{n,0}-x^*\| \\
&\le \mu_1(\|x_n-x^*\|)\|x_n-x^*\|.
\end{aligned}$$

Similarly, we get that

$$\begin{aligned}
\|w_{n,2}-x^*\| &\le \lambda(\|x_n-x^*\|)\|w_{n,1}-x^*\| \\
&\le \lambda^2(\|x_n-x^*\|)\|w_{n,0}-x^*\| \\
&\le \mu_2(\|x_n-x^*\|)\|x_n-x^*\|. \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\|w_{n,i}-x^*\| &\le \lambda^i(\|x_n-x^*\|)\|w_{n,0}-x^*\| \\
&\le \mu_i(\|x_n-x^*\|)\|x_n-x^*\|. \\
\|x_{n+1}-x^*\| &= \|w_{n,k}-x^*\| \\
&\le \lambda^k(\|x_n-x^*\|)\|w_{n,0}-x^*\| \\
&\le \mu_k(\|x_n-x^*\|)\|x_n-x^*\|.
\end{aligned}$$

That is we have $w_{n,i}\in U(x^*,r^*)$, $i=0,1,2,\ldots,k$ and

$$\|x_{n+1}-x^*\| \le \bar{c}\,\|x_n-x^*\|, \tag{34}$$

where $\bar{c}=\mu_k(\|x_n-x^*\|)\in[0,1)$, so $\lim_{n\to\infty}x_n=x^*$ and $x_{n+1}\in U(x^*,r^*)$. The uniqueness part of the proof can be found in [2]. □

## 3. Optimal computational efficiency

The discussion of computational efficiency is one of the important parts in the development of iterative methods. It is equally true that the construction of higher order method is important only if the method is efficient. Here, we will explore that what method (depending on the number of steps) of the $k$-step family (4) has better efficiency for solving systems of nonlinear equations in $\mathbb{R}^m$. The computational efficiency index (CEI) and the computational cost ($\mathcal{C}$) of an iterative method of convergence order $p$ are defined as ([15])

$$CEI(k,\mu_0,\mu_1,l,m)=p^{\frac{1}{\mathcal{C}(k,\mu_0,\mu_1,l,m)}} \tag{35}$$

and

$$\mathcal{C}(k,\mu_0,\mu_1,l,m)= P_0(k,m)\mu_0+P_1(m)\mu_1+P(k,m,l). \tag{36}$$

Here $P_0(k,m)$ represents the number of evaluations of scalar functions used in the evaluations of function $F=(f_1,f_2,.....,f_m)^T$ and divided difference $[y,x\,;F]$. The divided difference $[y,x;\,F]$

of $F$ is an $m \times m$ matrix with elements ([11])

$$[x, y;\ F]_{ij} = \frac{f_i(x_1, ....., x_j, y_{j+1}, ....., y_m) - f_i(x_1, ....., x_{j-1}, y_j, ....., y_m)}{x_j - y_j},\ \ 1 \leqslant i, j \leqslant m.$$

$P_1(m)$ is the number of evaluations of scalar functions of $F'$, i.e. $\frac{\partial f_i}{\partial x_j}$, $1 \leqslant i, j \leqslant m$; $P(k, m, l)$ represents the number of products or quotients needed per iteration. The quantities $\mu_0 > 0$ and $\mu_1 > 0$ are ratios between products and evaluations whereas $l \geqslant 1$ is the ratio between products and quotients. Such ratios are required to express the value of $\mathcal{C}(k, \mu_0, \mu_1, l, m)$ in terms of products. Without loss of generality, we will perform our analysis assuming an LU factorization to solve the associated linear systems. If we use the another type of approach, the analysis will be same.

To compute $F$ in any iterative method we calculate $m$ scalar functions and if we compute the divided difference $[y, x\,; F]$ then we evaluate $m(m-1)$ scalar functions, where $F(x)$ and $F(y)$ are computed separately. We must add $m^2$ quotients from any divided difference. The number of scalar evaluations is $m^2$ for any new derivative $F'$. In order to compute an inverse linear operator we solve a linear system, where we have $m(m-1)(2m-1)/6$ products and $m(m-1)/2$ quotients in the LU decomposition and $m(m-1)$ products and $m$ quotients in the resolution of two triangular linear systems. It is supposed that a quotient is equivalent to $l$ products. We should also add $m^2$ products for the multiplication of a matrix with a vector or of a matrix by a scalar and $m$ products for the multiplication of a vector by a scalar.

Keeping in view the above considerations, the various evaluations in the first step of (4) are: $P_0(1, m) = m(m+2)$, $P_1(m) = m^2$ and $P(1, m, l) = \frac{m}{6}\big(2m^2 + 21m - 5 + 3l(3m+5)\big)$. Then, after the $k$ steps, we have $P_0(k, m) = m(m+k+1)$, $P_1(m) = m^2$ and $P(k, m, l) = \frac{m}{6}\big(2m^2 + (18k+3)m + 3l(3m+4k+1) - 5\big)$. Finally, the computational efficiency index (CEI) is given as

$$CEI(k, \mu_0, \mu_1, l, m) = p^{\frac{1}{C}} = (2k+2)^{\frac{1}{Mk+N}}, \tag{37}$$

where $M = m\mu_0 + 3m^2 + 2ml$ and $N = (\mu_0 + \mu_1)m^2 + \frac{m}{6}(2m^2 + 3m + 3l(3m+1) - 5)$.

In order to compute the optimal point of CEI for the iterative methods of the family defined by (4), we consider $\frac{d}{dk}(\ln CEI) = 0$, then we get

$$\frac{d}{dk}\Big(\frac{1}{Mk+N}\ln(2k+2)\Big) = 0,$$

which implies that

$$\frac{1}{(Mk+N)^2}\left(M\ln(2k+2) - \frac{2(Mk+N)}{2k+2}\right) = 0,$$

that is

$$\frac{1}{2}(2k+2)\ln(2k+2) - k - \frac{(\mu_0+\mu_1)m + \frac{1}{6}(2m^2 + 3m + 3l(3m+1) - 5)}{\mu_0 + 3m + 2l} = 0. \tag{38}$$

For the given values of $m$, $\mu_0$, $\mu_1$ and $l$, we can solve Eq. (38) in terms of $k$, which is the value where the computational efficiency index (35) attains its optimal point. For an instance, if we have $m = 2$, $l = 2.33$, $\mu_0 = \mu_1 = 61.44$ then the solution of (38) yields $k = 2$, which shows that the two-step method of the scheme is better. Table 1 shows the values of $k$ as a function of $m$, $\mu_0$ and $\mu_1$ that are non-negative integer solutions of Eq. (38). The values of $m$ are displayed along first column. In all cases, the ratio $l$ is assumed as 2.33. The reason for taking $l = 2.33$ and $\mu_1 = 61.44$, 1/3, 1/5 will be clear in section 5.

Table 1(a). Values of $k$ for optimal efficiency.

| $m$ | $\mu_1 = 61.44$ | | | $\mu_1 = 1/3$ | | |
|---|---|---|---|---|---|---|
| (lr)2-4(lr)5-7 | $\mu_0 = \mu_1$ | $\mu_0 = 2\mu_1$ | $\mu_0 = 4\mu_1$ | $\mu_0 = 4\mu_1$ | $\mu_0 = 6\mu_1$ | $\mu_0 = 12\mu_1$ |
| 2 | 2 | **2** | 1 | 1 | 1 | 1 |
| 3 | 3 | 2 | 2 | 1 | 1 | 1 |
| 4 | 4 | 3 | 3 | 1 | 1 | 1 |
| 5 | 4 | 4 | 3 | 1 | 1 | 1 |
| 6 | 5 | 4 | 4 | 1 | 1 | 1 |
| 7 | 5 | 5 | 4 | 1 | 1 | 2 |
| 8 | 6 | 5 | 5 | 1 | 1 | 2 |
| 9 | 6 | 5 | 5 | 1 | 1 | **2** |
| 10 | 6 | 6 | 6 | 1 | 2 | 2 |
| 15 | 8 | 8 | 7 | 2 | 2 | 2 |
| 20 | 9 | 9 | 9 | 2 | 2 | 3 |

Table 1(b). Values of $k$ for optimal efficiency.

| $m$ | | $\mu_1 = 1/5$ | |
|---|---|---|---|
| (lr)2-4 | $\mu_0 = 10\mu_1$ | $\mu_0 = 15\mu_1$ | $\mu_0 = 20\mu_1$ |
| 2 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 |
| 4 | 1 | 1 | 1 |
| 5 | 1 | 1 | 1 |
| 6 | 1 | 1 | 1 |
| 7 | 1 | 1 | 1 |
| 8 | 1 | 1 | 2 |
| 9 | 1 | 2 | 2 |
| 10 | 2 | 2 | 2 |
| 15 | 2 | 2 | 2 |
| 20 | 2 | 2 | **3** |

## 4. Numerical examples

Here, we shall demonstrate the theoretical results of local convergence which we have proved in section 2. To do so, the methods of the family (4) of order four, six, eight and ten are chosen. Let us denote these methods by $M_4$, $M_6$, $M_8$ and $M_{10}$, respectively. The divided difference in the examples is defined by $[x, y\,; F] = \int_0^1 F'(y + \theta(x - y))d\theta$. We consider four numerical examples, which are presented as follows:

**Example 4.1.** [2] Suppose that the motion of an object in three dimensions is governed by system of differential equations

$$\begin{aligned} &f_1'(x) - f_1(x) - 1 = 0,\\ &f_2'(y) - (e-1)y - 1 = 0,\\ &f_3'(z) - 1 = 0, \end{aligned}$$

with $x$, $y$, $z \in \Omega$ for $f_1(0) = f_2(0) = f_3(0) = 0$. Then, the solution of the system is given for $u = (x, y, z)^T$ by function $F := (f_1, f_2, f_3) : \Omega \to \mathbb{R}^3$ defined by

$$F(u) = \Big(e^x - 1, \frac{e-1}{2}y^2 + y, z\Big)^T.$$

The Fréchet-derivative is given by

$$F'(u) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then for $x^* = (0,0,0)^T$, we deduce that $\omega_0(t) = (e-1)t$, $\omega(t) = e^{\frac{1}{e-1}}t$ and $v_0(t) = v(t) = e^{\frac{1}{e-1}}$. The calculated values of parameters are displayed in Table 2.

Table 2. Numerical results for example 1.

| $M_4$ | $M_6$ | $M_8$ | $M_{10}$ |
|---|---|---|---|
| $r_1 = 0.382733$ | $r_1 = 0.382733$ | $r_1 = 0.382733$ | $r_1 = 0.382733$ |
| $r^{(1)} = 0.0693497$ | $r^{(2)} = 0.0085753$ | $r^{(3)} = 0.0008313$ | $r^{(4)} = 0.0000762427$ |
| $r^* = 0.0693497$ | $r^* = 0.0085753$ | $r^* = 0.0008313$ | $r^* = 0.0000762427$ |

So, from Theorem 2.2. it follows that the above considered methods of the scheme (4) converge to $x^*$ and remain in $\bar{U}(x^*, r^*)$.

**Example 4.2.** [2] Let $B_1 = B_2 = C[0,1]$ be the space of continuous functions defined on the interval $[0,1]$ and be equipped with max norm. Let $\Omega = \bar{U}(0,1)$. Define function $F$ on $\Omega$ by

$$F(\varphi)(x) = \phi(x) - 10\int_0^1 x\theta\varphi(\theta)^3 d\theta.$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 30\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \quad for \quad each \quad \xi \in \Omega.$$

Then for $x^* = 0$ we have that $\omega_0(t) = 15t$, $\omega(t) = 30t$ and $v_0(t) = v(t) = 2$. Numerical results are displayed in Table 3. Theorem 2.2. guarantees the convergence of methods to $x^* = 0$ provided that $x_0 \in \bar{U}(x^*, r^*)$.

Table 3. Numerical results for example 2.

| $M_4$ | $M_6$ | $M_8$ | $M_{10}$ |
|---|---|---|---|
| $r_1 = 0.033333$ | $r_1 = 0.033333$ | $r_1 = 0.033333$ | $r_1 = 0.033333$ |
| $r^{(1)} = 0.0048334$ | $r^{(2)} = 0.00052943$ | $r^{(3)} = 0.000049826$ | $r^{(4)} = 4.5506 \times 10^{-6}$ |
| $r^* = 0.0048334$ | $r^* = 0.00052943$ | $r^* = 0.000049826$ | $r^* = 4.5506 \times 10^{-6}$ |

**Example 4.3.** Let us consider the function $F := (f_1, f_2, f_3) : \Omega \to \mathbb{R}^3$ defined by

$$F(x) = \big(10\,x_1 + \sin(x_1+x_2) - 1,\ 8\,x_2 - \cos^2(x_3 - x_2) - 1,\ 12\,x_3 + \sin(x_3) - 1\big)^T, \qquad (39)$$

where $x = (x_1,\, x_2,\, x_3)^T$.

The Fréchet-derivative is given by

$$F'(x) = \begin{bmatrix} 10+\cos(x_1+x_2) & \cos(x_1+x_2) & 0 \\ 0 & 8+\sin 2(x_2-x_3) & -2\sin(x_2-x_3) \\ 0 & 0 & 12+\cos(x_3) \end{bmatrix}.$$

Then for the solution $x^* = \{0.0689\ldots, 0.2464\ldots, 0.0769\ldots\}^T$, we have that $\omega_0(t) = \omega(t) = 0.269812t$, $v_0(t) = 2$ and $v(t) = 13.0377$. Thus, the computed values of parameters are displayed in Table 4.

Table 4. Numerical results for example 3.

| $M_4$ | $M_6$ | $M_8$ | $M_{10}$ |
|---|---|---|---|
| $r_1 = 2.470856$ | $r_1 = 2.470856$ | $r_1 = 2.470856$ | $r_1 = 2.470856$ |
| $r^{(1)} = 0.102483$ | $r^{(2)} = 0.00168791$ | $r^{(3)} = 0.00000255622$ | $r^{(4)} = 3.86223 \times 10^{-7}$ |
| $r^* = 0.102483$ | $r^* = 0.00168791$ | $r^* = 0.00000255622$ | $r^* = 3.86223 \times 10^{-7}$ |

**Example 4.4.** The Vander Waal equation of state for a vapor is ([12])

$$\Big(P + \frac{a}{V^2}\Big)(V - b) = RT, \tag{40}$$

where $P$ is the pressure ($Pa = N/m^2$), $V$ is specific volume ($m^3/kg$), $T$ is the temperature ($K$), $R$ is the gas constant ($J/kgK$) and $a$ and $b$ are emperical constants. Consider water vapor, for which $R = 461.495J/kgK$, $a = 1703.28Pa(m^3/kg)$ and $b = 0.00169099(m^3/kg)$. Equation (40) can be rearranged into the form

$$PV^3 - (Pb + RT)V^2 + aV - ab = 0. \tag{41}$$

The solution of this problem is $x^* = 0.337822\ldots$. Then, we have $\omega_0(t) = \omega(t) = 17.5162t$, $v_0(t) = 2$ and $v(t) = 8.75857$. The numerical results are shown in Table 5.

Table 5. Numerical results for example 4.

| $M_4$ | $M_6$ | $M_8$ | $M_{10}$ |
|---|---|---|---|
| $r_1 = 0.038060$ | $r_1 = 0.038060$ | $r_1 = 0.038060$ | $r_1 = 0.038060$ |
| $r^{(1)} = 0.00224643$ | $r^{(2)} = 5.6 \times 10^{-5}$ | $r^{(3)} = 1.270 \times 10^{-6}$ | $r^{(4)} = 2.83 \times 10^{-8}$ |
| $r^* = 0.00224643$ | $r^* = 5.6 \times 10^{-5}$ | $r^* = 1.270 \times 10^{-6}$ | $r^* = 2.83 \times 10^{-8}$ |

## 5. Applications

The methods $M_4$, $M_6$, $M_8$ and $M_{10}$ corresponding to $k = 1, 2, 3, 4$ of the proposed $k$-step scheme (4) are applied to solve different systems of nonlinear equations in $\mathbb{R}^m$. The numerical computations listed in the following tables are performed on computer algebra system such as *Mathematica* [22] with multi-precision arithmetic. We record the number of iterations ($n$) needed to converge to a solution such that

$$\|x_{n+1} - x_n\| + \|F(x_n)\| < 10^{-200} \quad \text{(stopping criterion)}.$$

To confirm theoretical order of convergence, the computational order ($p_c$) is computed using the formula

$$p_c = \frac{\ln(\|x_{n+1} - x_n\|/\|x_n - x_{n-1}\|)}{\ln(\|x_n - x_{n-1}\|/\|x_{n-1} - x_{n-2}\|)}, \tag{42}$$

([5]) taking into consideration the last four approximations in the iterative process. In the comparison of performance of methods, we also include the real CPU time elapsed during the execution of program which is computed by *Mathematica* command "TimedUsed[ ]".

According to the definition of the computational cost (35), an estimation of the factors $\mu_0$ and $\mu_1$ are claimed. In order to do this, we express the cost of the evaluation of elementary functions in terms of products, which depend on computing machine, software and computer arithmetics used ([9]). In Table 6, the elapsed CPU time (measured in milliseconds) in the computation of elementary functions and an estimation of the cost of the elementary functions in product units are displayed. The programs are performed in the processor, AMD A8-7410 APU with AMU Radeon R5 Graphics @ 2.20 GHz(64 bit Operating System) Microsoft Window

10 Ultimate 2016 and are complied by *Mathematica* 9.0 using multi-precision arithmetics. It can be observed from Table 6 that for this hardware and the software, the computational cost of quotient with respect to multiplication is, $l = 2.33$.

Table 6. Estimation of computational cost of elementary functions for $x = \sqrt{3} - 1$ and $y = \sqrt{5}$.

| Arithmetics | x∗ y | x/y | $\sqrt{x}$ | exp(x) | ln(x) | sin(x) | cos(x) | arccos(x) | arctan(x) |
|---|---|---|---|---|---|---|---|---|---|
| CPU time | 0.021875 | 0.05096 | 0.02656 | 2.2765 | 1.4640 | 1.5547 | 1.54531 | 2.4046 | 2.4125 |
| Cost | 1 | 2.33 | 1.214 | 104.071 | 66.928 | 71.072 | 70.642 | 109.93 | 110.29 |

Numerical results displayed in Tables 7–9 show:

– The number $k$, used for number of steps a method consists of.
– The method $M_{2k+2}$ $(1 \leq k \leq 4)$, where $2k+2$ is the order of convergence.
– The required number of iterations $(n)$.
– The error $||x_{n+1} - x_n||$ of approximation to the corresponding solution of considered problems, wherein $a(-h)$ denotes $a \times 10^{-h}$.
– The computational order of convergence $(p_c)$.
– The computational cost $\mathcal{C}$ in terms of products.
– The computational efficiency index CEI.
– The elapsed CPU time (CPU-time) in seconds.

We test the convergence behavior of the family of iterative methods (4), for $1 \leq k \leq 4$, on the following three problems:

**Problem 5.1.** Consider the system of two equations (selected from [21]):

$$x_1 + e^{x_2} - \cos x_2 = 0,$$
$$3x_1 - \sin x_1 - x_2 = 0,$$

with the initial value $x_0 = (-1, 1)^T$ towards the root $x^* = (0, 0)^T$. For this problem the corresponding values of parameters $\mu_0$ and $\mu_1$, calculated by using the Table 6, are 123.39 and 61.44. This shows that $\mu_0 \simeq 2\mu_1$. Note that the other parameters are $(m, l) = (2, 2.33)$. These values are used for computing computational costs and efficiency indices, and also to verify the results of optimal computational efficiency of section 3. The numerical results are shown in Table 7. It can be observed that the two-step method $M_6$ gives a maximum value of CEI and a minimum value of CPU-time (Table 7).

Table 7. Numerical results for problem 1, where $(m, l, \mu_0, \mu_1) = (2, 2.33, 123.39, 61.44)$.

| $k$ | Method | $n$ | $\|x_{n+1} - x_n\|$ | $p_c$ | $\mathcal{C}$ | CEI | CPU-time |
|---|---|---|---|---|---|---|---|
| 1 | $M_4$ | 6 | 3.69(−470) | 4.000 | 1027 | 1.0013508 | 0.7214 |
| 2 | $M_6$ | 5 | 3.61(−537) | 6.000 | 1295 | 1.0013846 | 0.6564 |
| 3 | $M_8$ | 4 | 3.01(−203) | 8.000 | 1563 | 1.0013313 | 0.7338 |
| 4 | $M_{10}$ | 4 | 2.72(−385) | 10.000 | 1831 | 1.0012583 | 0.7404 |

**Problem 5.2.** The boundary value problem ([14])

$$u'' + u^3 = 0, \quad u(0) = 0, \quad u(1) = 1,$$

is studied. Consider the following partitioning of the interval $[0, 1]$:

$$t_0 = 0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = 1, \quad t_{j+1} = t_j + h, \quad h = 1/l.$$

Let us define $u_0 = u(t_0) = 0$, $u_1 = u(t_1), ..., u_{l-1} = u(t_{l-1})$, $u_l = u(t_l) = 1$. If we discretize the problem by using the numerical formula for second derivative

$$u''_k = \frac{u_{k-1} - 2u_k + u_{k+1}}{h^2}, \ \ (k = 1, 2, 3, \dots, l-1),$$

we obtain a system of $l - 1$ nonlinear equations in $l - 1$ variables:

$$u_{k-1} - 2u_k + u_{k+1} + h^2 u_k^3 = 0, \ \ (k = 1, 2, 3, \dots, l-1).$$

In particular, we solve this problem for $l = 10$ so that $m = 9$ by selecting $u_0 = \{1, 1, \overset{9-times}{\cdots}, 1\}^T$ as the initial value. The corresponding solution is given by $x^* = \{0.1055\dots, 0.2110\dots, 0.3165\dots, 0.4216\dots, 0.5259\dots, 0.6289\dots, 0.7293\dots, 0.8258\dots, 0.9167\dots\}^T$. In this case $(\mu_0, \mu_1) = (4, 1/3)$, which implies that $\mu_0 = 12\mu_1$. Numerical results are shown in Table 8. Note that the two-step method $M_6$ gives a maximum value of CEI and a minimum value of CPU-time.

Table 8. Numerical results for problem 2, where $(m, l, \mu_0, \mu_1) = (9, 2.33, 4, 1/3)$.

| $k$ | Method | $n$ | $\|x_{n+1} - x_n\|$ | $p_c$ | $\mathcal{C}$ | CEI | CPU-time |
|---|---|---|---|---|---|---|---|
| 1 | $M_4$ | 6 | 1.69(−536) | 4.000 | 1242 | 1.0011168 | 0.2506 |
| 2 | $M_6$ | 5 | 1.72(−576) | 6.000 | 1562 | 1.0011478 | 0.2356 |
| 3 | $M_8$ | 4 | 2.22(−210) | 8.000 | 1883 | 1.0011049 | 0.2647 |
| 4 | $M_{10}$ | 4 | 5.21(−392) | 10.000 | 2204 | 1.0010453 | 0.2730 |

**Problem 5.3.** Now considering the mixed Hammerstein integral equation ([14]):

$$x(s) = 1 + \frac{1}{5}\int_0^1 G(s,t)x(t)^3 dt,$$

where $x \in C[0, 1]$; $s, t \in [0, 1]$ and the kernel $G$ is

$$G(s,t) = \begin{cases} (1-s)t, \ t \le s, \\ s(1-t), \ s \le t. \end{cases}$$

We transform the above equation into a finite-dimensional problem by using Gauss-Legendre quadrature formula given as

$$\int_0^1 f(t)dt \approx \sum_{j=1}^{m} \varpi_j f(t_j),$$

where the abscissas $t_j$ and the weights $\varpi_j$ are determined for $m = 20$ by Gauss-Legendre quadrature formula. Denoting approximation of $x(t_i)$ by $x_i$ $(i = 1, 2, \dots, 20)$, we obtain the system of nonlinear equations

$$5x_i - 5 - \sum_{j=1}^{20} a_{ij} x_j^3 = 0, \ \ i = 1, 2, \dots, 20,$$

where

$$a_{ij} = \begin{cases} \varpi_j t_j (1 - t_i), \text{ if } j \le i, \\ \varpi_j t_i (1 - t_j), \text{ if } \ i < j. \end{cases}$$

Initial approximation chosen for this problem is $x_0 = \{\frac{3}{2}, \frac{3}{2}, \overset{20-times}{\cdots}, \frac{3}{2}\}^T$ for obtaining the solution:

$$x^* = \{1.00037\ldots, 1.00191\ldots, 1.00455\ldots, 1.00805\ldots, 1.012099\ldots, 1.019030\ldots, 1.02490\ldots, 1.03064\ldots, 1.035712\ldots, 1.03956\ldots, 1.04169\ldots, 1.04177\ldots, 1.03965\ldots, 1.03545\ldots, 1.04143\ldots, 1.02266\ldots, 1.01549\ldots, 1.00894\ldots, 1.00194\ldots, 1.00037\ldots\}^T.$$

The calculated values of parameters $\mu_0$ and $\mu_1$ are 4 and 1/5, respectively. So, we have that $\mu_0 = 20\mu_1$. Numerical results of this problem are displayed in Table 9, which show that the three-step method $M_8$ possesses maximum efficiency and minimum CPU-time.

Table 9. Numerical results for problem 3, where $(m, l, \mu_0, \mu_1) = (20, 2.33, 4, 1/5)$.

| $k$ | Method | $n$ | $\|x_{n+1} - x_n\|$ | $p_c$ | $\mathcal{C}$ | CEI | CPU-time |
|---|---|---|---|---|---|---|---|
| 1 | $M_4$ | 5 | 4.04(−243) | 4.000 | 7324 | 1.0001893 | 3.5512 |
| 2 | $M_6$ | 5 | 1.89(−1142) | 6.000 | 8698 | 1.0002060 | 3.4386 |
| 3 | $M_8$ | 4 | 8.48(−436) | 8.000 | 10071 | 1.0002065 | 3.1849 |
| 4 | $M_{10}$ | 4 | 4.40(−835) | 10.000 | 11444 | 1.0002012 | 3.4820 |

From the numerical results, we can observe that all the considered methods of the general scheme (4) show consistent convergence behavior. Calculated value of the computational order of convergence ($p_c$) of each method verifies the theoretical order of convergence proved in Section 2. The results of optimal computational efficiency obtained in Table 1 are also verified in the considered problems by calculating computational efficiency index of the methods as displayed in the penultimate columns of Tables 7–9. This fact is highlighted by using the bold number for $k$-value in Table 1. Thus, the values of $k$ for optimal efficiency of iterative methods of the scheme (4) in Table 1 are interesting not only from theoretical point of view but are also useful in practice. Notice that the parameters $\mu_0$ and $\mu_1$ in Table 1 are selected according to their computed values in problems 5.1., 5.2. and 5.3. From the numerical values of efficiency index (CEI) and elapsed CPU time (CPU-time), we can observe that the method with large efficiency uses less computing time than the method with small efficiency. This shows that the efficiency results are in complete agreement with the CPU time utilized in the execution of program.

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