TWMS J. Pure Appl. Math., V.14, N.2, 2023, pp.162-177
DOI: 10.30546/2219-1259.14.2.2023.162

IDENTIFICATION PROBLEM FOR NONLINEAR SCHRÖDINGER TYPE EQUATION

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ABSTRACT. The identification problem of determining of a several unknown coefficients of the Schrödinger type equation is considered, correctness of this problem is studied and necessary condition for the solution is found. For simplicity, the basic presentation of this problem is considered for case of one-dimensional nonlinear Schrödinger type equation with boundary observation.

Keywords: Schrödinger type equation, identification problem, quantum processes, ill-posedness, variational method.

AMS Subject Classification: 35Q55, 49N45.

1. Introduction

Schrödinger type equation is generalized of the well - known standard quantum mechanical Schrödinger equation [12, 14]. In works [1-3, 9, 11, 15, 17] are developed the methods of solution of Schrödinger type equation. This type equation arises in adoptive optics [20], atomistic - molecular computational simulations, studies of bifurcation phenomena for nonlinear models etc. Studying of the processes of wave propagation in a nonlinear medium is one of the pressing problems of nonlinear optics and this problem occurs in various fields of modern physics and technology. In work [16], the problem of determination of the complex quantum potential by observation and quality criterion other then in this work is studied this problem different then studying below. Let $\psi(x,t)$- complex wave function at the point with space coordinate x and a time t. For a large range of applications, of nonlinear optics equation describing the slow variation of the function $\psi(x,t)$ in a dissipative one-dimensional medium with a quadratic dispersion has form [21]:

$$i\frac{\partial \psi}{\partial t} + r_2(x,t,\psi)\frac{\partial^2 \psi}{\partial x^2} + r_1(x,t,\psi)\frac{\partial \psi}{\partial x} + r_0(x,t,\psi)\psi = 0, \tag{1}$$

where $i=\sqrt{-1}$, ψ is a wave function, the coefficients $r_j(x,t,\psi)$, $j=0,1,2$, are the functions of (x,t), which described the heterogeneity of the medium of propagation and their dependence on the function $\psi=\psi(x,t)$ is its nonlinear properties. A special case of equation (1) is the following nonlinear Schrödinger type equation, which often arises in applications:

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Manuscript received 2019.

$$i\rho^2\frac{\partial\psi}{\partial t}+\frac{\partial}{\partial x}\left(a_0\left(x,t\right)\frac{\partial\psi}{\partial x}\right)-a\left(x,t\right)\psi+a_1\left|\psi\right|^2\psi=f(x,t), \tag{2}$$

where the functions $a_0\left(x,t\right), a\left(x,t\right)$ describe the heterogeneity property of the medium with a quadratic dispersion, ρ, a_1- a real numbers, but these may be functions of $(x,$ t). In practices are faced identification problem about determination of the coefficients $a_0\left(x,t\right)$, $a\left(x,t\right)$ and the complex wavefunction $\psi\left(x,t\right)$ of nonlinear Schrödinger type are faced identification problem about determination of the coefficients $a_0\left(x,t\right)$, $a\left(x,t\right)$ and the complex wavefunction $\psi\left(x,t\right)$ of nonlinear Schrödinger type equation (2). However, there exist a few investigations about determination the height order coefficients of nonlinear Schrödinger type equation [4, 6] (see also [8, 21, 15] etc.). Below in detail considered the problem of determination of the coefficient of the highest derivative and potential terms in the nonlinear Schrödinger type equation (2). These problems are typical and contain the theoretical and practical interest. The unknown coefficients investigated for in the class of measurable and bounded functions. The investigation of this problem based on the methodology of works [9, 15, 22].

2. Statement of the problem

Let us $l>0, T>0-$ are given numbers; $0\le x\le l, \quad 0\le t\le T, \quad \Omega_t=(0,l)\times(0,t), \quad \Omega=\Omega_T$; $L_p\left(0,l\right)$- Lebesgue space of measurable functions on $(0,l)$, which are integrable with degree $p\ge 1$; $C^k\left(\left[0,T\right],B\right)-$ Banach space of $k\ge 0-$ times continuously differentiable on $[0,T]$ functions with values belonging to the Banach space B; $W_p^k\left(0,l\right), \quad W_p^{k,m}\left(\Omega\right)$ are Sobolev spaces of functions with generalized derivatives of order $k\ge 0-$ respect to the variable x and of the order $m\ge 0-$ respect to variable t, which are integrable with degree $p\ge 1$; $W_\infty^1\left(0,l\right)\equiv\left\{w: w\in L_\infty\left(0,l\right), \frac{dw}{dx}\in L_\infty\left(0,l\right)\right\}$- Banach space with indicating properties; the symbol $\overset{0}{\forall}$ indicates that this property have place for almost all values of a variable. Below, everywhere positive constant, independent of the estimated quantities will denoted by $c_j, \quad j=0,1,2,...$

Consider the initial-boundary value problem for the nonlinear Schrödinger type equation:

$$i\rho^2\frac{\partial\psi}{\partial t}+\frac{\partial}{\partial x}\left(v_0\left(x\right)\frac{\partial\psi}{\partial x}\right)-v_1\left(x\right)\psi+a_1\left|\psi\right|^2\psi=f\left(x,t\right), \quad (x,t)\in\Omega, \tag{3}$$

$$\psi\left(x,0\right)=\varphi\left(x\right), \quad x\in\left(0,l\right), \tag{4}$$

$$\frac{\partial\psi\left(0,t\right)}{\partial x}=\frac{\partial\psi\left(l,t\right)}{\partial x}=0, \quad t\in\left(0,T\right), \tag{5}$$

where ρ, a_1 - are the real valued numbers, $\varphi\left(x\right)$, $f\left(x,t\right)-$ given complex-valued measurable functions, satisfied the following conditions:

$$\varphi\in W_2^2\left(0,l\right), \quad \frac{d\varphi\left(0\right)}{dx}=\frac{d\varphi\left(l\right)}{dx}=0, \quad f\in W_2^{0,1}\left(\Omega\right), \tag{6}$$

coefficients $v_0\left(x\right), v_1\left(x\right)$ are unknown real-valued functions. Our aim is to identify the function $\left\{v_0\left(x\right), \quad v_1\left(x\right), \quad \psi\left(x,t\right)\right\}$ from conditions (3) -(5) if additionally given the following boundary observations:

$$\psi\left(0,t\right)=y_0\left(t\right), \quad \psi\left(l,t\right)=y_1\left(t\right), \quad t\in\left(0,T\right), \tag{7}$$

where $y_0=y_0\left(t\right), \quad y_1=y_1\left(t\right)$ are given complex-valued functions from $L_2\left(0,T\right)$.

For the solution of the considering problem, we will apply the regularizing procedure as below [18] (see also [8-10, 13, 15, 19, 20, 22]). The vector function $v = v(x) = (v_0(x), v_1(x))$, we will consider for on the set:

$$V \equiv \Big\{ v = (v_0, v_1) : v_0 \in W_2^1(0,l), v_1 \in L_2(0,l), 0 < b_0 \le v_0(x) \le b_1,$$

$$\left| \frac{dv_0(x)}{dx} \right| \le b_2, 0 < b_4 \le v_1(x) \le b_3, \overset{0}{\forall} x \in (0,l) \Big\},$$

where $b_j > 0, j = \overline{0,4}$- given numbers. We will call this set as a set of admissible coefficients. Consider the variational problem about determination of the vector-function $\{v(x)\}$ by minimization of the functional:

$$J_\alpha(v) = \beta_0 \|\psi(0,.) - y_0\|^2_{L_2(0,T)} + \beta_1 \|\psi(l,.) - y_1\|^2_{L_2(0,T)} + \alpha \|\nu - \omega\|^2_H \tag{8}$$

on the set V under the conditions (3)-(7), where $\beta_0 \ge 0, \beta_1 \ge 0-$ given numbers, such that $\beta_0 + \beta_1 \ne 0$, element $\omega \in H \equiv W_2^1(0,l) \times L_2(0,l)$, is given, $\alpha \ge 0-$given number. We must remark that, in the identification problems with data's given no exact, than number $\alpha \ge 0-$ determined from "residual principle" (see: [18]) in class of positive numbers. The problem of determining of the function $\psi = \psi(x,t) \equiv \psi(x,t;v)$ from the conditions (3)-(5) for each $v \in V$, we will call the direct problem. Solution of this problem is the function $\psi = \psi(x,t) \equiv \psi(x,t;v)$ from the space $B_0 \equiv C^0\left([0,T], W_2^2(0,l)\right) \cap C^1([0,T], L_2(0,l))$, which satisfies the equation (3) for any $t \in [0,T]$ and almost all $x \in (0,l)$, the conditions (4), (5) satisfies for almost all $x \in (0,l)$ and $t \in (0,T)$, respectively. The problem of minimizing of the functional $\mathrm{J}_\alpha(v)$ on the set V under the conditions (3) -(7), we will call the identification problem (8).

The direct problem is the initial-value problem for nonlinear Schrödinger equation. This problem studied in the works [1-3, 8, 9, 11, 15, 17, 22] etc. From these results, follows that the direct problem (3)-(5) for each $v \in V$, under shown above conditions has a unique solution in the space B_0 and for this solution hold a priory estimate

$$\|\psi(.,t)\|_{W_2^2(0,l)} + \left\| \frac{\partial \psi(.,t)}{\partial t} \right\|_{L_2(0,l)} \le c_0 \left(\|\varphi\|_{W_2^2(0,l)} + \|f\|_{W_2^{0,1}(\Omega)} \right). \tag{9}$$

3. The correctness of the identification problem

Theorem 3.1. *The identification problem (8) has at least of one solution.*

Proof. Take any minimizing sequence $\left\{v^k\right\} \in V$:

$$\lim_{k \to \infty} J_\alpha\left(v^k\right) = J_\alpha^* = \inf_{v \in V} J_\alpha(v). \tag{10}$$

Let $\psi_k = \psi_k(x,t) \equiv \psi\left(x,t;v^k\right), k = 1,2,...$. Here element v^k for each k belongs to V. Then from results of [1-3, 8, 9, 11, 15, 17, 22] follows, that the direct problem (3) -(5) for each k has a unique solution from B_0 and hold the estimate (9):

$$\|\psi_k(.,t)\|_{W_2^2(0,l)} + \left\| \frac{\partial \psi_k(.,t)}{\partial t} \right\|_{L_2(0,l)} \le c_1, \forall t \in [0,T], \quad k = 1,2,... \tag{11}$$

where $c_1 > 0$ denotes the right-hand side of (9) and is independent of k. Since the set V is a closed, bounded and convex subset of $B \equiv W_\infty^1(0,l) \times L_\infty(0,l)$, from the sequence $\left\{v^k\right\}$, we can extract a subsequence, which for simplicity is again denoted by $\{v^k\}$, that $v_0^k \to v_0 \quad (*)$ weakly in $L_\infty(0,l)$,

$$\begin{aligned} &\tfrac{dv_0^k}{dx} \to \tfrac{dv_0}{dx} \quad (*) \ \textit{weakly} \ L_\infty(0,l), \\ &v_1^k \to v_1 \quad (*) \ \textit{weakly} \ L_\infty(0,l), \quad \textit{at} \quad k \to \infty. \end{aligned} \tag{12}$$

In addition, by virtue of the structure of V, it is easy to believe that it is $(*)$weakly closed set, so $v \in V$. From the limiting relations and embedding $W_\infty^1(0,l)$ to the space $L_\infty(0,l)$, we obtain that $v_0^k \to v_0$ strongly in $L_\infty(0,l)$ at $k \to \infty$. From estimate (11) is follows that the sequence $\{\psi_k(x,t)\}$ uniformly bounded in the norm B_0. Then, from this sequence, by using the equation (3), may be extract a subsequence, which for simplicity is again denoted by $\{\psi_k(x,t)\}$, that $\psi_k(.,t) \to \psi(.,t)$ weakly in $W_2^2(0,l)$, $\frac{\partial \psi_k(.,t)}{\partial t} \to \frac{\partial \psi(.,t)}{\partial t}$ weakly in $L_2(0,l)$ for each $t \in [0,T]$ at $k \to \infty$. Due the compactness of the embedding the space B_0 to $C^0\left([0,T], W_2^1(0,l)\right)$ (see: [11], p.214), we have:

$$\|\psi_k(.,t) - \psi(.,t)\|_{W_2^1(0,l)} \to 0 \qquad k \to \infty \tag{13}$$

uniformly with respect to $t \in [0,T]$. It is clear verify that for every element of the sequence $\{\psi_k(x,t)\} \in B_0$ satisfies the identity

$$\begin{aligned} &\int_0^l [i\rho^2 \tfrac{\partial \psi_k(x,t)}{\partial t} + \tfrac{\partial}{\partial x}\left(v_0^k(x)\tfrac{\partial \psi_k(x,t)}{\partial x}\right) - v_1^k(x)\psi_k(x,t) \\ &+a_1|\psi_k(x,t)|^2\psi_k(x,t) - f(x,t)]\bar{\eta}(x)\,dx = 0, \quad k = 1,2,... \end{aligned} \tag{14}$$

for $\forall t \in [0,T]$ and arbitrary function $\eta \in L_2(0,l)$, holds the initial and boundary conditions

$$\psi_k(x,0) = \varphi(x), \quad \overset{0}{\forall} x \in (0,l), \quad k = 1,2,..., \tag{15}$$

$$\frac{\partial \psi_k(0,t)}{\partial x} = \frac{\partial \psi_k(l,t)}{\partial x} = 0, \quad \overset{0}{\forall} t \in (0,T), \quad k = 1,2,.... \tag{16}$$

It is easy to prove the validity of following limit relations:

$$\lim_{k\to\infty} \int_0^l \frac{\partial}{\partial x}\left(v_0^k(x)\frac{\partial \psi_k(x,t)}{\partial x}\right)\bar{\eta}(x)\,dx = \int_0^l \frac{\partial}{\partial x}\left(v_0(x)\frac{\partial \psi(x,t)}{\partial x}\right)\bar{\eta}(x)\,dx, \tag{17}$$

$$\lim_{k\to\infty} \int_0^l v_1^k(x)\psi_k(x,t)\bar{\eta}(x)\,dx = \int_0^l v_1(x)\psi(x,t)\bar{\eta}(x)\,dx, \tag{18}$$

$$\lim_{k\to\infty} \int_0^l |\psi_k(x,t)|^2\psi_k(x,t)\bar{\eta}(x)\,dx = \int_0^l |\psi(x,t)|^2\psi(x,t)\bar{\eta}(x)\,dx \tag{19}$$

for each $t \in [0,T]$ and for any function $\eta \in L_2(0,l)$ at $k \to \infty$. Thus, by using the limit relations (17) -(19), if the pass to the limit in the integral identity (14), then from there, we get the following identity:

$$\int_0^l (i\rho^2 \tfrac{\partial\psi(x,t)}{\partial t} + \tfrac{\partial}{\partial x}\left(v_0(x)\tfrac{\partial\psi(x,t)}{\partial x}\right) - v_1(x)\psi(x,t)$$

$$+a_1 |\psi(x,t)|^2 \psi(x,t) - f(x,t))\bar\eta(x)\,dx = 0$$

for each $t \in [0,T]$ and for any function $\eta \in L_2(0,l)$. It follows that the limit function $\psi(x,t)$ satisfies the equation (3) for each $t \in [0,T]$ and for almost all$x \in (0,l)$. Using the limiting relation (13) at $t = 0$, we get:

$$\|\psi_k(.,0) - \psi(.,0)\|_{L_2(0,l)} \to 0 \qquad k \to \infty.$$

Using this relation and make transition to the limit in the inequality.

$$\|\psi(.,0) - \varphi\|_{L_2(0,l)} \le \|\psi(.,0) - \psi_k(.,0)\|_{L_2(0,l)} + \|\psi_k(.,0) - \varphi\|_{L_2(0,l)}$$

we obtain that the limit function $\psi(x,t)$ satisfies the initial condition (4) for almost all $x \in (0,l)$. Finally, we will prove that the limit function $\psi(x,t)$ satisfies the boundary conditions (5). For functions from space B_0 hold the limit relations:

$$\frac{\partial\psi_k(s,.)}{\partial x} \to \frac{\partial\psi(s,.)}{\partial x} \quad \text{weakly in} \quad L_2(0,T) \quad \text{at} \quad k \to \infty \tag{20}$$

for $s = 0$ and $s = l$. Then, by using this limit relation and boundary conditions (16), after passing to the limit $k \to \infty$, in the equality:

$$\int_0^T \frac{\partial\psi(s,t)}{\partial x}\bar\eta(t)\,dt = \int_0^T \left(\frac{\partial\psi(s,t)}{\partial x} - \frac{\partial\psi_k(s,t)}{\partial x}\right)\bar\eta(t)\,dt + \int_0^T \frac{\partial\psi_k(s,t)}{\partial x}\bar\eta(t)\,dt, \quad s = 0,l,$$

we found:

$$\int_0^T \frac{\partial\psi(s,t)}{\partial x}\bar\eta(t)\,dt = 0, \quad s = 0,l$$

for any function $\bar\eta = \bar\eta(t)$ from $L_2(0,T)$. From these equalities follows that the limit function $\psi(x,t)$ satisfies the boundary conditions (5) for almost all $t \in (0,T)$. Thus, we have proved that the limit function $\psi(x,t)$ of sequence $\{\psi_k(x,t)\}$ is the solution of direct problem (10) - (12), corresponding to the limit function $v = v(x)$ belonging to V, that is $\psi = \psi(x,t) \equiv \psi(x,t;v)$. In addition, the limit function $\psi(x,t)$ is the transition to the weak limit of a weakly convergent subsequence $\{\psi_k(x,t)\}$ to the function $\psi(x,t)$ for each $t \in [0,T]$.

From the compact embedding of B_0 to space $C^0([0,l], L_2(0,T))$, we have:

$$\|\psi_k(x,.) - \psi(x,.)\|_{L_2(0,T)} \to 0$$

for every $x \in [0,l]$ at $k \to \infty$. From this fact for $x = 0, x = l$ and from the weak lower semi continuity of the norm in the spaces $L_2(0,T)$ and non-negativity of the numbers β_0, β_1, α, we obtain:

$$J_\alpha^* \le J_\alpha(v) \le \underline{\lim}_{k\to\infty} J_\alpha\left(v^k\right) = \inf_{v\in V} J_\alpha(v) = J_\alpha^*.$$

It follows that $v = v(x) \in V$ and $v = v(x)$ is a solution of the problem of identification (8). Theorem 3.1 is proved. □

Now we introduce the following set of admissible coefficients:

$$V_1 \equiv \left\{ v = (v_0, v_1) : v_0 \in W_2^2(0,l), v_1 \in W_2^1(0,l), \quad 0 < b_0 \le v_0(x) \le b_1, \left| \frac{dv_0(x)}{dx} \right| \le b_2, \right.$$

$$\left. \left| \frac{d^2 v_0(x)}{dx^2} \right| \le b_3, 0 < \tilde{b}_4 \le v_1(x) \le b_4, \left| \frac{dv_1(x)}{dx} \right| \le b_5, \forall x \in \overset{0}{(0,l)} \right\}, \tag{21}$$

where $b_j > 0, j = \overline{0,5}, \tilde{b}_4 > 0$- given numbers. Below, we consider the problem of identification about minimizing the functional (8) on the set V_1 under conditions (3) -(5). Let $B \equiv W_\infty^2(0,l) \times W_\infty^1(0,l)$, $B_1 \equiv W_2^2(0,l) \times W_2^1(0,l)$ and for functions $\varphi(x)$, $f(x,t)$ additionally to above shown, satisfies the following conditions:

$$\frac{d^3\varphi}{dx^3} \in L_2(0,l), \frac{\partial f}{\partial x} \in W_2^{0,1}(\Omega). \tag{22}$$

In the works [9, 15], the authors have shown that, for each $v = v(x)$ from the set V_1, for solution of the problem (3) -(5) hold the estimate

$$\begin{aligned} &\|\psi(.,t)\|_{W_2^3(0,l)} + \left\| \tfrac{\partial \psi(.,t)}{\partial t} \right\|_{L_2(0,l)} + \left\| \tfrac{\partial^2 \psi(.,t)}{\partial x \partial t} \right\|_{L_2(0,l)}, \\ &c_2 \left(\|\varphi\|_{W_2^3(0,l)} + \|f\|_{W_2^{0,1}(\Omega)} + \left\| \tfrac{\partial f}{\partial x} \right\|_{W_2^{0,1}(\Omega)} \right), \forall t \in [0,T]. \end{aligned} \tag{23}$$

Easy verify that Theorem 1 holds also in the set V_1 for all $\alpha \ge 0$. But, examples similarly from [9, 15] shows that for $\alpha = 0$, the solution of problem (8) in the sets V and V$_1$ is nonunique and nonstable. However, in the case $\alpha > 0$ hold.

Theorem 3.2. *There exist a dense subset G of space H such that for any $\omega \in G$ and $\alpha > 0$ identification problem about minimization of the functional (8) in the set V_1 has a unique solution.*

Proof. We approve that the functional

$$J_0(v) = \beta_0 \|\psi(0,.) - y_0\|_{L_2(0,T)}^2 + \beta_1 \|\psi(l,.) - y_1\|_{L_2(0,T)}^2 \tag{24}$$

is continuous on the set V_1. Let $\delta v \in B$ is increment of a element $v \in V_1$ such, that $v + \delta v \in V$. Denote $\delta\psi = \delta\psi(x,t) \equiv \psi(x,t;v+\delta v) - \psi(x,t;v)$, where $\psi(x,t;v)$ is solution of the direct problem (3) -(5) for $v \in V_1$. From conditions (3) -(5) is follows that the function $\delta\psi = \delta\psi(x,t)$ is a solution of the following initial-boundary value problem:

$$\begin{aligned} &i\rho^2 \tfrac{\partial \delta\psi}{\partial t} + \tfrac{\partial}{\partial x} \left((v_0(x) + \delta v_0(x)) \tfrac{\partial \delta\psi}{\partial x} \right) - (v_1(x) + \delta v_1(x)) \delta\psi \\ &+ a_1 \left(|\psi_\delta|^2 + |\psi|^2 \right) \delta\psi + a_1 \psi_\delta \psi \delta\bar{\psi} = -\tfrac{\partial}{\partial x} \left(\delta v_0(x) \tfrac{\partial \psi}{\partial x} \right) + \delta v_1(x)\psi, \quad (x,t) \in \Omega \end{aligned} \tag{25}$$

$$\delta\psi(x,0) = 0, \quad x \in (0,l), \tag{26}$$

$$\frac{\partial \delta\psi(0,t)}{\partial x} = \frac{\partial \delta\psi(l,t)}{\partial x} = 0, \quad t \in (0,T), \tag{27}$$

where $\psi_\delta = \psi_\delta(x,t) \equiv \psi(x,t;v+\delta v)$. For estimating of the solution of this initial-boundary value problem, both sides of equation (25) we multiplied by a function $\delta\bar{\psi}(x,t)$ and integrate the result on the Ω_t. After extracting the resulting equality, with its complex conjugate, by using the initial condition (26), we have:

$$\|\rho\delta\psi\left(.,t\right)\|_{L_2(0,l)}^2 = 2\alpha_1\int\limits_{\Omega_t} Im(\dot{\psi}\psi_\delta(\delta\bar{\psi})^2)dxd\tau$$

$$+2\int\limits_{\Omega_t} Im\left[\left(-\frac{\partial}{\partial x}\left(\delta v_0\left(x\right)\frac{\partial\psi}{\partial x}\right)+\delta v_1\left(x\right)\psi\right)\delta\bar{\psi}\right]dxd\tau,$$

Hence, applying the Cauchy-Schwarz inequality, we obtain:

$$\|\rho\delta\psi\left(.,t\right)\|_{L_2(0,l)}^2 \le 2\left|a_1\right|\int\limits_{\Omega_t}\left|\psi_\delta\right|\left|\psi\right|\left|\delta\psi\right|^2 dxd\tau$$
$$+\int\limits_{\Omega_t}\left|-\tfrac{\partial}{\partial x}\left(\delta v_0\left(x\right)\tfrac{\partial\psi}{\partial x}\right)+\delta v_1\left(x\right)\psi\right|^2 dxd\tau+\int\limits_{\Omega_t}\left|\delta\psi\right|^2 dxd\tau, \quad \forall t\in[0,T]. \tag{28}$$

It is well known that, if function $\phi\left(.,t\right)\in W_2^1\left(0,l\right)$,$\forall t\in[0,T]$, is not equal to zero at the endpoints of segment $[0,l]$, then hold the following inequality (see.[11], p.80, remark 2.1):

$$\|\phi\left(.,t\right)\|_{L_\infty(0,l)} \le \beta\left\|\frac{\partial\phi\left(.,t\right)}{\partial x}\right\|_{L_2(0,l)}^{\frac{1}{2}}\|\phi\left(.,t\right)\|_{L_2(0,l)}^{\frac{1}{2}}+\tilde{\beta}\left\|\phi\right\|_{L_2(0,l)}, \tag{29}$$

where $\beta>0,\tilde{\beta}>0$- are some constants. From this inequality and (23), we get

$$\|\psi\left(.,t\right)\|_{L_\infty(0,l)} \le c_4, \quad \|\psi_\delta\left(.,t\right)\|_{L_\infty(0,l)} \le c. \tag{30}$$

If we take into account of these inequalities in (28), then we get:

$$\|\delta\psi\left(.,t\right)\|_{L_2(0,l)}^2 \le \left(2\left|a_1\right|c_4^2+1\right)\int\limits_0^t\|\delta\psi\left(.,\tau\right)\|_{L_2(0,l)}^2 d\tau$$
$$+\int\limits_{\Omega_t}\left|-\tfrac{\partial}{\partial x}\left(\delta v_0\left(x\right)\tfrac{\partial\psi}{\partial x}\right)+\delta v_1\left(x\right)\psi\right|^2 dxd\tau, \quad \forall t\in\left[0,T\right]. \tag{31}$$

By estimating the left term of this inequality, we have:

$$\int\limits_{\Omega_t}\left|-\frac{\partial}{\partial x}\left(\delta v_0\left(x\right)\frac{\partial\psi}{\partial x}\right)+\delta v_1\left(x\right)\psi\right|^2 dxd\tau \le c_5\int\limits_0^t\|\psi\left(.,\tau\right)\|_{W{}^0{}_2^2(0,l)}^2 d\tau\,\|\delta v\|_B^2, \quad \forall t\in\left[0,T\right].$$

Taking into account of this inequality and the inequalities (23), (28), we get:

$$\|\delta\psi\left(.,t\right)\|_{L_2(0,l)}^2 \le c_6\int\limits_0^t\|\delta\psi\left(.,\tau\right)\|_{L_2(0,l)}^2 d\tau + c_7\left\|\delta v\right\|_B^2, \quad \forall t\in\left[0,T\right].$$

Applying the Gronwall lemma to this inequality, we have:

$$\|\delta\psi\left(.,t\right)\|_{L_2(0,l)}^2 \le c_8\left\|\delta v\right\|_B^2, \quad \forall t\in\left[0,T\right]. \tag{32}$$

We now estimate the $\frac{\partial\delta\psi}{\partial x}$. For this aim we multiply both sides of equation (25) by a function $L\left(\delta\bar{\psi}\right) = -\frac{\partial}{\partial x}\left(\left(v_0\left(x\right)+\delta v_0\left(x\right)\right)\frac{\partial\delta\bar{\psi}}{\partial x}\right)$ and integrate the resulting equation on the Ω_t, then we get the following equality:

$$\int\limits_{\Omega_t} \left(i\rho^2 \tfrac{\partial\delta\psi}{\partial t} L\left(\delta\bar{\psi}\right) - \left|L\left(\delta\psi\right)\right|^2 - \left(v_1\left(x\right) + \delta v_1\left(x\right)\right)\delta\psi L\left(\delta\bar{\psi}\right)\right.$$

$$\left.+a_1\left(\left|\psi_\delta\right|^2 + \left|\psi\right|^2\right)\delta\psi L\left(\delta\bar{\psi}\right) + a_1\psi_\delta\psi\delta\bar{\psi}L\left(\delta\bar{\psi}\right)\right)dxd\tau \tag{33}$$

$$= -\int\limits_{\Omega_t}\left(\tfrac{\partial}{\partial x}\left(\delta v_0\left(x\right)\tfrac{\partial\psi}{\partial x}\right) - \delta v_1\left(x\right)\psi\right)L\left(\delta\bar{\psi}\right)dxd\tau, \quad \forall t\in\left[0,T\right].$$

By applying integration by parts with respect to x of the left and right hand -sides of this equation, then from the resulting equality by extracting its complex conjugate, we get:

$$\int\limits_{\Omega_t}\rho^2\tfrac{\partial}{\partial t}\left(\left(v_0\left(x\right)+\delta v_0\left(x\right)\right)\left|\tfrac{\partial\delta\psi}{\partial x}\right|^2\right)dxd\tau$$

$$= 2\int\limits_{\Omega_t} Im\left(\left(v_0\left(x\right)+\delta v_0\left(x\right)\right)\left(\tfrac{dv_1(x)}{dx}+\tfrac{d\delta v_1(x)}{dx}\right)\delta\psi\tfrac{\partial\delta\bar{\psi}}{\partial x}\right)dxd\tau$$

$$-2a_1\int\limits_{\Omega_t} Im\left(\left(v_0\left(x\right)+\delta v_0\left(x\right)\right)\tfrac{\partial}{\partial x}\left(\left|\psi_\delta\right|^2+\left|\psi\right|^2\right)\delta\psi\tfrac{\partial\delta\bar{\psi}}{\partial x}\right)dxd\tau$$

$$-2a_1\int\limits_{\Omega_t} Im\left(\left(v_0\left(x\right)+\delta v_0\left(x\right)\right)\tfrac{\partial}{\partial x}\left(\psi_\delta\psi\right)\delta\bar{\psi}\tfrac{\partial\delta\bar{\psi}}{\partial x}\right)dxd\tau$$

$$-2a_1\int\limits_{\Omega_t} Im\left(\left(v_0\left(x\right)+\delta v_0\left(x\right)\right)\left(\psi_\delta\psi\right)\left(\tfrac{\partial\delta\bar{\psi}}{\partial x}\right)^2\right)dxd\tau$$

$$-2\int\limits_{\Omega_t} Im\left(\left(v_0\left(x\right)+\delta v_0\left(x\right)\right)\tfrac{\partial}{\partial x}\left[\tfrac{\partial}{\partial x}\left(\delta v_0\left(x\right)\tfrac{\partial\psi}{\partial x}\right)-\delta v_1\left(x\right)\psi\right]\tfrac{\partial\delta\bar{\psi}}{\partial x}\right)dxd\tau, \quad \forall t\in\left[0,T\right]. \tag{34}$$

If take into account the conditions $v+\delta v\in V$, $\delta\psi\left(x,0\right)=0, x\in\left(0,l\right)$, then from the Cauchy-Schwarz inequality and the inequalities (30), we obtain:

$$b_0\left\|\frac{\partial\delta\psi\left(.,t\right)}{\partial x}\right\|^2_{L_2(0,l)} \le c_9\int\limits_0^t\left\|\frac{\partial\delta\psi\left(.,\tau\right)}{\partial x}\right\|^2_{L_2(0,l)}d\tau + c_{10}\int\limits_{\Omega_t}\left|\delta\psi\right|^2dxd\tau$$

$$+4b_1\int\limits_{\Omega_t}\left|\delta v_0\left(x\right)\right|^2\left|\frac{\partial^3\psi}{\partial x^3}\right|^2dxd\tau + 8b_1 + \int\limits_{\Omega_t}\left|\frac{d\delta v_0\left(x\right)}{dx}\right|^2\left|\frac{\partial^2\psi}{\partial x^2}\right|^2dxd\tau$$

$$+4b_1\int\limits_{\Omega_t}\left|\tfrac{d\delta v_1(x)}{dx}\right|^2\left|\psi\right|^2dxd\tau + 4b_1\int\limits_{\Omega_t}\left(\left|\tfrac{d^2\delta v_0(x)}{dx^2}\right|+\left|\delta v_1\left(x\right)\right|\right)^2\left|\tfrac{\partial\psi}{\partial x}\right|^2dxd\tau$$

$$+c_8\left|a_1\right|\left(2+b_1\right)\int\limits_{\Omega_t}\left(\left|\frac{\partial\psi_\delta}{\partial x}\right|+\left|\frac{\partial\psi}{\partial x}\right|\right)^2\left|\delta\psi\right|^2dxd\tau, \quad \forall t\in\left[0,T\right], \tag{35}$$

where $c_9 = b_1\left(3\left|a_1\right|c_4+\left|a_1\right|c_4^2+1+b_5\right)$, $c_{10} = b_1\left(\left|a_1\right|c_4^2+b_5\right)$. Now, we must estimate the last addend in right hand side of this inequality. It is known that for arbitrary function $\phi\left(.,t\right)\in W\,0_2^1\left(0,l\right)$,$\forall t\in\left[0,T\right]$ hold the following inequality (see.[11], pp.79-80):

$$\left\|\phi\left(.,t\right)\right\|^2_{L_\infty(0,l)} \le \beta\left\|\frac{\partial\phi\left(.,t\right)}{\partial x}\right\|_{L_2(0,l)}\left\|\phi\left(.,t\right)\right\|_{L_2(0,l)}, \quad \forall t\in\left[0,T\right], \tag{36}$$

where $\beta > 0$- is constant. In this inequality, instead of the function $\phi(x,t)$ take the functions $\frac{\partial\psi_\delta(x,t)}{\partial x}$ and $\frac{\partial\psi(x,t)}{\partial x}$. Then, from (36) for functions $\psi_\delta(x,t)$, $\psi(x,t)$, we can write the inequalities:

$$\left\|\frac{\partial\psi(.,t)}{\partial x}\right\|_{L_\infty(0,l)} \le c_{11}, \quad \left\|\frac{\partial\psi_\delta(.,t)}{\partial x}\right\|_{L_\infty(0,l)} \le c_{11}. \tag{37}$$

Using the inequalities and taking into account the (23), (30), (35), we have:

$$\left\|\frac{\partial\delta\psi(.,t)}{\partial x}\right\|^2_{L_2(0,l)} \le c_{12}\int_0^t \left\|\frac{\partial\delta\psi(.,\tau)}{\partial x}\right\|^2_{L_2(0,l)} d\tau + c_{13}\|\delta v\|^2_B, \quad \forall t\in[0,T].$$

Applying to this inequality the Gronwall lemma, we obtain the estimate:

$$\left\|\frac{\partial\delta\psi(.,t)}{\partial x}\right\|^2_{L_2(0,l)} \le c_{14}\|\delta v\|^2_B, \quad \forall t\in[0,T].$$

Summing this estimate with (32), we have:

$$\|\delta\psi(.,t)\|^2_{W_2^1(0,l)} \le c_{15}\|\delta v\|^2_B, \quad \forall t\in[0,T]. \tag{38}$$

Now, consider the increment of the functional $J_0(v)$ on any element $v\in V_1$. From (24), we have:

$$\begin{aligned}\delta J_0(v) = J_0(v+\delta v) - J_0(v) = 2\beta_0\int_0^T Re\left[(\psi(0,t)-y_0(t))\,\delta\bar{\psi}(0,t)\right]dt \\ +2\beta_1\int_0^T Re\left[(\psi(l,t)-y_1(t))\,\delta\bar{\psi}(l,t)\right]dt \\ +\beta_0\|\delta\psi(0,.)\|^2_{L_2(0,T)} + \beta_1\|\delta\psi(l,.)\|^2_{L_2(0,T)}\end{aligned} \tag{39}$$

From the well-known embedding theorem for the trace of functions from $W_2^{1,0}(\Omega)$ (see. [11]), we have:

$$\|\psi(0,.)\|^2_{L_2(0,T)} + \|\psi(l,.)\|^2_{L_2(0,T)} \le c_{16}\|\psi\|^2_{W_2^{1,0}(\Omega)}, \tag{40}$$

$$\|\delta\psi(0,.)\|^2_{L_2(0,T)} + \|\delta\psi(l,.)\|^2_{L_2(0,T)} \le c_{16}\|\delta\psi\|^2_{W_2^{1,0}(\Omega)}. \tag{41}$$

Then, from these inequalities, taking account of (26), (41), we obtain:

$$\|\psi(0,.)\|^2_{L_2(0,T)} + \|\psi(l,.)\|^2_{L_2(0,T)} \le c_{17}, \tag{42}$$

$$\|\delta\psi(0,.)\|^2_{L_2(0,T)} + \|\delta\psi(l,.)\|^2_{L_2(0,T)} \le c_{18}\|\delta v\|^2_B. \tag{43}$$

Applying the Cauchy-Schwarz inequality and using (42), (43), as well as the condition $y_0, y_1 \in L_2(0,l)$, to expiration (39), then we obtain the estimate:

$$|\delta J_0(v)| \le c_{19}\left(\|\delta v\|_B + \|\delta v\|^2_B\right), \forall v\in V_1.$$

This implies that the functional $J_0(v)$ is continuous on the set V_1. Moreover, $J_0(v)\ge 0, \forall v\in V_1$. The set V_1 is a closed, bounded and convex in the space H. A space H as a Hilbert space is uniformly convex space. Consequently, all the conditions of the theorem from [5, 7] about existence and uniqueness of solution of the problem (8) are satisfied. Then, there is exist a dense subset of G the space H such that for any $\omega \in G$ and $\alpha > 0$ identification problem (8) has a unique solution. Theorem 3.2 is proved. □

4. Necessary condition for the solution

Let us the function $\Phi = \Phi(x,t)$ is a solution of the following dual problem:

$$i\rho^2 \frac{\partial \Phi}{\partial t} + \frac{\partial}{\partial x}\left(v_0(x)\frac{\partial \Phi}{\partial x}\right) - v_1(x)\Phi + 2a_1|\psi|^2\Phi + a_1\psi^2\bar{\Phi} = 0, \quad (x,t) \in \Omega, \tag{44}$$

$$\Phi(x,T) = 0, \quad x \in (0,l), \tag{45}$$

$$\frac{\partial \Phi(0,t)}{\partial x} = -2\frac{\beta_0}{v_0(0)}(\psi(0,t) - y_0(t)), \quad t \in (0,T), \tag{46}$$

$$\frac{\partial \Phi(l,t)}{\partial x} = 2\frac{\beta_1}{v_0(l)}(\psi(l,t) - y_1(t)), \quad t \in (0,T), \tag{47}$$

where $\psi = \psi(x,t) \equiv \psi(x,t;v)$- solution of direct problem (3) -(5) for $v \in V_1$. Under solution of the dual problem (44) -(47), we mean a function $\Phi = \Phi(x,t)$ from the space $C^0\left([0,T], W_2^1(0,l)\right)$, which satisfies the identity:

$$\int\limits_{\Omega}\left[\left(-i\rho^2\Phi\frac{\partial \bar{\eta}_1}{\partial t} - v_0(x)\frac{\partial \Phi}{\partial x}\frac{\partial \bar{\eta}_1}{\partial x} - v_1(x)\Phi\bar{\eta}_1 + 2a_1|\psi|^2\Phi\bar{\eta}_1\right) + a_1\psi^2\bar{\Phi}\bar{\eta}_1\right]dxdt$$

$$= -2\beta_0\int\limits_0^T(\psi(0,t) - y_0(t))\bar{\eta}_1(0,t)\,dt - 2\beta_1\int\limits_0^T(\psi(l,t) - y_1(t))\bar{\eta}_1(l,t)\,dt \tag{48}$$

for any function $\eta_1 = \eta_1(x,t)$ from the space $W_2^{1,1}(\Omega)$, satisfied the condition $\eta_1(x,0) = 0, x \in (0,l)$. Follows the technique of reducing the initial-boundary value problem with inhomogeneous boundary conditions to homogeneous in [11], in the works [9, 15] proved that, if the functions $\varphi(x)$, $f(x,t)$ satisfies conditions (6), (25) and $y_0, y_1 \in W_2^1(0,T)$, when the dual problem (44) -(47) for each $v \in V_1$ has a unique solution in the space $C^0\left([0,T], W_2^1(0,l)\right)$ and for this solution hold the estimate

$$\|\Phi(.,t)\|_{W_2^1(0,l)} \le c_{20}\left(\|\psi(0,.) - y_0\|_{W_2^1(0,T)} + \|\psi(l,.) - y_1\|_{W_2^1(0,T)}\right), \quad \forall t \in [0,T]. \tag{49}$$

Now, we consider the increment of the functional $J_\alpha(v)$ on the set V_1. Using the (8) and (39), we have

$$\delta J_\alpha(v) = J_\alpha(v+\delta v) - J_\alpha(v) = 2\beta_0\int\limits_0^T Re\left[(\psi(0,t) - y_0(t))\,\delta\bar{\psi}(0,t)\right]dt$$

$$+2\beta_1\int\limits_0^T Re\left[(\psi(l,t) - y_1(t))\,\delta\bar{\psi}(l,t)\right]dt$$

$$+\beta_0\|\delta\psi(0,.)\|^2_{L_2(0,T)} + \beta_1\|\delta\psi(l,.)\|^2_{L_2(0,T)}$$

$$+2\alpha\int_0^l\left[\left(v_0(x)-\omega_0(x)\right)\delta v_0(x)+\left(\frac{dv_0(x)}{dx}-\frac{d\omega_0(x)}{dx}\right)\frac{d\delta v_0(x)}{dx}\right]dx$$

$$+2\alpha\int_0^l\left[\left(\frac{d^2v_0(x)}{dx^2}-\frac{d^2\omega_0(x)}{dx^2}\right)\frac{d^2\delta v_0(x)}{dx^2}\right]dx$$

$$+2\alpha\int_0^l\left[\left(v_1(x)-\omega_1(x)\right)\delta v_1(x)+\left(\frac{dv_1(x)}{dx}-\frac{d\omega_1(x)}{dx}\right)\frac{d\delta v_1(x)}{dx}\right]dx+\alpha\left\|\delta v\right\|_H^2, \tag{50}$$

where $\delta v\in B\equiv W_\infty^2(0,l)\times W_\infty^1(0,l)$- increment of the element $v=v(x)$ from the set V_1 such that $v+\delta v\in V_1$, and the function $\delta\psi=\delta\psi(x,t)\equiv\psi(x,t;v+\delta v)-\psi(x,t;v)$ is a solution of the initial-boundary value problem (25) -(27). We transform the sum of the first two terms on the right side of (50). For this aim, we proof that the following equality is holds:

$$2\beta_0\int_0^T Re\left[\left(\psi(0,t)-y_0(t)\right)\delta\bar{\psi}(0,t)\right]dt+2\beta_1\int_0^T Re\left[\left(\psi(l,t)-y_1(t)\right)\delta\bar{\psi}(l,t)\right]dt$$

$$=-\int_\Omega\left[Re\left(\frac{\partial\psi(x,t)}{\partial x}\frac{\partial\bar{\Phi}(x,t)}{\partial x}\right)\delta v_0(x)+Re\left(\psi(x,t)\,\bar{\Phi}(x,t)\right)\delta v_1(x)\right]dxdt+R_1(\delta v), \tag{51}$$

where the residual term $R_1(\delta v)$ determined by the formula:

$$R_1(\delta v)=-\int_\Omega\left[Re\left(\frac{\partial\delta\psi(x,t)}{\partial x}\frac{\partial\bar{\Phi}(x,t)}{\partial x}\right)\delta v_0(x)+Re\left(\delta\psi(x,t)\,\bar{\Phi}(x,t)\right)\delta v_1(x)\right]dxdt$$

$$+\int_\Omega\left[Re\left(2a_1\left|\delta\psi\right|^2\psi\bar{\Phi}\right)+Re\left(\left(\delta\psi\right)^2\bar{\psi}\bar{\Phi}\right)+Re\left(a_1\left|\delta\psi\right|^2\delta\psi\bar{\Phi}\right)\right]dxdt. \tag{52}$$

Indeed, from the fact that the solution of the direct problem belongs to B for solution of initial boundary value problem (25) -(27), we can write the following integral identity:

$$\int_\Omega\left(i\rho^2\frac{\partial\delta\psi}{\partial t}+\frac{\partial}{\partial x}\left(\left(v_0(x)+\delta v_0(x)\right)\frac{\partial\delta\psi}{\partial x}\right)-\left(v_1(x)+\delta v_1(x)\right)\delta\psi\right.$$

$$\left.+a_1\left(\left|\psi_\delta\right|^2+\left|\psi\right|^2\right)\delta\psi++a_1\psi_\delta\psi\delta\bar{\psi}\right)\bar{\eta}dxdt$$

$$=-\int_\Omega\left(\frac{\partial}{\partial x}\left(\delta v_0(x)\frac{\partial\psi}{\partial x}\right)-\delta v_1(x)\,\psi\right)\bar{\eta}(x,t)dxdt$$

for any function $\eta=\eta(x,t)$ from $L_2(\Omega)$. In this integral identity instead of $\eta(x,t)$ from $L_2(\Omega)$ take the function $\Phi(x,t)$ from $C^0\left([0,T],W_2^1(0,l)\right)$. Then, using integration by parts in the first term on the right side, in the second term on the left side, we have:

$$\int_\Omega\left(i\rho^2\frac{\partial\delta\psi}{\partial t}\bar{\Phi}-\left(v_0(x)+\delta v_0(x)\right)\frac{\partial\delta\psi}{\partial x}\frac{\partial\bar{\Phi}}{\partial x}-\left(v_1(x)+\delta v_1(x)\right)\delta\psi\bar{\Phi}\right.$$

$$\left.+a_1\left(\left|\psi_\delta\right|^2+\left|\psi\right|^2\right)\delta\psi\bar{\Phi}+a_1\psi_\delta\psi\delta\bar{\psi}\bar{\Phi}\right)dxdt$$

$$= \int_{\Omega} \left(\delta v_0(x) \frac{\partial \psi}{\partial x} \frac{\partial \bar{\Phi}}{\partial x} + \delta v_1(x) \psi \bar{\Phi} \right) dxdt. \tag{53}$$

Now, in the integral identity (48) instead of $\eta_1(x,t)$ from the space $W_2^{1,1}(\Omega)$ take the function $\delta\psi(x,t)$ from space B_0. Then, by extracting from (53), complex conjugation of the resulting equation, we get:

$$2\beta_0 \int_0^T \left(\bar{\psi}(0,t) - \bar{y}_0(t)\right) \delta\psi(0,t)dt + 2\beta_1 \int_0^T \left(\bar{\psi}(l,t) - \bar{y}_1(t)\right) \delta\psi(l,t)dt$$

$$+ \int_{\Omega} \left(\delta v_0(x) \frac{\partial \psi}{\partial x} \frac{\partial \bar{\Phi}}{\partial x} + \delta v_1(x) \psi \bar{\Phi} \right) dxdt$$

$$= \int_{\Omega} \left(-\delta v_0(x) \frac{\partial \delta\psi}{\partial x} \frac{\partial \bar{\Phi}}{\partial x} - \delta v_1(x) \delta\psi \bar{\Phi} \right) dxdt$$

$$+ \int_{\Omega} \left(a_1 \psi_\delta \psi \delta\bar{\psi} \bar{\Phi} - a_1 \left(\bar{\psi}\right)^2 \delta\psi \Phi \right) dxdt + \int_{\Omega} \left(a_1 |\psi_\delta|^2 \delta\psi \bar{\Phi} - a_1 |\psi|^2 \delta\psi \bar{\Phi} \right) dxdt. \tag{54}$$

If we use here the equalities: $|\psi_\delta|^2 \delta\psi\bar{\Phi} - |\psi|^2 \delta\psi\bar{\Phi} = |\delta\psi|^2 \psi\bar{\Phi} + (\delta\psi)^2 \bar{\psi}\bar{\Phi} + |\delta\psi|^2 \delta\psi\bar{\Phi}$,

$$\psi_\delta \psi \delta\bar{\psi}\bar{\Phi} - \left(\bar{\psi}\right)^2 \delta\psi\Phi = (\psi)^2 \delta\bar{\psi}\bar{\Phi} + |\delta\psi|^2 \psi\bar{\Phi} - \left(\bar{\psi}\right)^2 \delta\psi\Phi$$

and summing the resulting equality with its complex conjugate, we have:

$$2\beta_0 \int_0^T Re\left[\left(\psi(0,t) - y_0(t)\right) \delta\bar{\psi}(0,t)\right] dt + 2\beta_1 \int_0^T Re\left[\left(\psi(l,t) - y_1(t)\right) \delta\bar{\psi}(l,t)\right] dt$$

$$= -\int_{\Omega} \left[Re\left(\frac{\partial \psi(x,t)}{\partial x} \frac{\partial \bar{\Phi}(x,t)}{\partial x} \right) \delta v_0(x) + Re\left(\psi(x,t)\bar{\Phi}(x,t)\right) \delta v_1(x) \right] dxdt$$

$$+ \int_{\Omega} Re\left(\delta\psi(x,t)\bar{\Phi}(x,t)\right) \delta v_1(x)\, dxdt +$$

$$+ \int_{\Omega} \left[Re\left(2a_1 |\delta\psi|^2 \psi\bar{\Phi}\right) + Re\left((\delta\psi)^2 \bar{\psi}\bar{\Phi}\right) + Re\left(a_1 |\delta\psi|^2 \delta\psi\bar{\Phi}\right) \right] dxdt.$$

Taking into account the notation $R_1(\delta v)$, we obtain proof of the equality (51).

Theorem 4.1. *Under the above shown conditions, for the first variation of the functional $J_\alpha(v)$ hold the following expression:*

$$\delta J_\alpha(v,w) = -\int_{\Omega} \left[Re\left(\frac{\partial \psi(x,t)}{\partial x} \frac{\partial \bar{\Phi}(x,t)}{\partial x} \right) w_0(x) + Re\left(\psi(x,t)\bar{\Phi}(x,t)\right) w_1(x) \right] dxdt$$

$$+2\alpha \int_0^l \left[\left(v_0(x) - \omega_0(x)\right) w_0(x) + \left(\frac{dv_0(x)}{dx} - \frac{d\omega_0(x)}{dx} \right) \frac{dw_0(x)}{dx} \right] dx$$

$$+2\alpha\int\limits_0^l\left[\left(\frac{d^2v_0(x)}{dx^2}-\frac{d^2\omega_0(x)}{dx^2}\right)\frac{d^2w_0(x)}{dx^2}\right]dx+$$

$$+2\alpha\int\limits_0^l\left[\left(v_1(x)-\omega_1(x)\right)w_1(x)+\left(\frac{dv_1(x)}{dx}-\frac{d\omega_1(x)}{dx}\right)\frac{dw_1(x)}{dx}\right]dx, \tag{55}$$

where $\psi=\psi(x,t)\equiv\psi(x,t;v)$- is solution of direct problem (3)-(7), $\Phi=\Phi(x,t)\equiv\Phi(x,t;v)$-is solution of the dual problem (44) -(47) for $v\in V_1$.

Proof. By using the (50) and (51), the increment of the functional $J_\alpha(v)$ on the element $v\in V_1$, can be represented as:

$$\Delta J_\alpha(v)=-\int\limits_\Omega\left[Re\left(\frac{\partial\psi(x,t)}{\partial x}\frac{\partial\bar{\Phi}(x,t)}{\partial x}\right)\delta v_0(x)+Re\left(\psi(x,t)\bar{\Phi}(x,t)\right)\delta v_1(x)\right]dxdt$$

$$+2\alpha\int\limits_0^l\left[\left(v_0(x)-\omega_0(x)\right)\delta v_0(x)+\left(\frac{dv_0(x)}{dx}-\frac{d\omega_0(x)}{dx}\right)\frac{d\delta v_0(x)}{dx}\right]dx$$

$$+2\alpha\int\limits_0^l\left[\left(\frac{d^2v_0(x)}{dx^2}-\frac{d^2\omega_0(x)}{dx^2}\right)\frac{d^2\delta v_0(x)}{dx^2}\right]dx$$

$$+2\alpha\int\limits_0^l\left[\left(v_1(x)-\omega_1(x)\right)\delta v_1(x)+\left(\frac{dv_1(x)}{dx}-\frac{d\omega_1(x)}{dx}\right)\frac{d\delta v_1(x)}{dx}\right]dx+R(\delta v), \tag{56}$$

where $R(\delta v)$ determined by the formula:

$$R(\delta v)=R_1(\delta v)+\beta_0\|\delta\psi(0,.)\|^2_{L_2(0,T)}+\beta_1\|\delta\psi(l,.)\|^2_{L_2(0,T)}+\alpha\|\delta v\|^2_H \tag{57}$$

and $R_1(\delta v)$ is in (52). For estimating of $R(\delta v)$, we used (52) and (57):

$$\alpha\|\delta v\|^2_H+|R(\delta v)|\le\beta_0\|\delta\psi(0,.)\|^2_{L_2(0,T)}+\beta_1\|\delta\psi(l,.)\|^2_{L_2(0,T)}$$

$$+\int\limits_\Omega\left(\left|\frac{\partial\delta\psi}{\partial x}\right|\left|\frac{\partial\Phi}{\partial x}\right||\delta v_0(x)|+|\delta\psi||\Phi||\delta v_1(x)|\right)dxdt$$

$$+3|a_1|\int\limits_\Omega|\delta\psi|^2|\psi||\Phi|dxdt+|a_1|\int\limits_\Omega|\delta\psi|^3|\Phi|dxdt. \tag{58}$$

If we take $\phi(x,t)=\delta\psi(x,t)$ in (32), then from (58), we can obtain the following estimate:

$$\|\delta\psi\|^2_{L_\infty(\Omega)}\le c_{21}\|\delta v\|^2_B. \tag{59}$$

Consequently, in the view of this estimate and estimates (26), (41), (49), (59), as well as, in conditions $y_0,y_1\in W_2^1(0,T)$ from (58) follows the validity of the inequality:

$$|R(\delta v)|\le c_{22}\|\delta v\|^2_B. \tag{60}$$

The means of this estimate that the residual term $R(\delta v)$ is an infinitesimal of higher order with respect to $\|\delta v\|_B$. Therefore, the first variation of the functional $J_\alpha(v)$ is in the form (55). Theorem 4.1 is proved. $\square$

Theorem 4.2. *Under the above shown conditions for the solution of the problem (8) hold the necessary condition in form of the following variational inequality*

$$\delta J_\alpha(\nu^*, \nu - \nu^*) \geq 0, \tag{61}$$

where the expression of the $\delta J_\alpha(\nu^*, \nu - \nu^*)$ *shown in (55)*

Proof. Necessary conditions for the solutions of extremely problems type (8) in the form of a variational inequality was studied in [19,20], see also [9,15]. Denote by

$$V_1^* \equiv \left\{ v^* \in V_1 : J_\alpha(v^*) = J_{\alpha*} = \inf_{v \in V_1} J_\alpha(v) \right\}$$

the set of solutions in the set V_1 of the identification problem (8). It is easy analogically [19, 20] to proof, that under the above shown conditions, the first variation of the functional $J_\alpha(\nu)$ is continuously in the set $v^* \in V_1^* \subset V_1$. Then as shown in [20], satisfies all conditions for holding a necessary condition in the form variation inequality (61) for the solution of identification problem (8). Theorem 4.2 is proved. $\square$

5. Conclusion

Studying above the identification problem developed the methodology of the works [8, 9, 15, 22]. In (8), we of the considered the problem about determination of several coefficients, especially the higher derivatives of a nonlinear Schrödinger type equation and obtained results are more general, then in previous works. Everywhere instead of equations (2) or (3), one can take a general Schrödinger type equation [11, 12, 14], here the results have the same formulation above getting for problem (8).

For solving the inverse problems applied the regularization method, which have an optimization or iterative nature [10-14, 18-20]. In well-known software packages for solution of the problems, such as Maple, Matlab, Ansys, Matcad and others, in addition to optimization and differential equations blocks, is recommended used of visualization result, which makes their application more attractively. Nonlinearity, discontinuity of the coefficients of the main equation and instability of the solution of the considering inverse problems are the main reasons for their complexity, slowdown of the computational procedure and decrease in the accuracy of the results.

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