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REVERSE PARABOLIC SOURCE IDENTIFICATION PROBLEM

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Abstract. The aim of the present work is to investigate the source identification problem for a reverse parabolic equation. The existence and uniqueness of the solution are established, along with stability estimates. These theoretical results are applied to proof of well-posedness of three specific source identification problems for the reverse parabolic partial differential equations.

Keywords: inverse problem, source identification problem, well-posedness, stability estimate, reverse parabolic equation.

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1. Introduction

Source identification problems (SIPs) are of crucial importance in the mathematical modeling of real-world processes (see [22, 34]). A comprehensive overview and classification of recent research on SIPs and nonlocal direct problems for partial differential equations (PDEs) are available in the literature [1-9, 11-17, 19, 21, 23-35, 37]. Note that article [19] is devoted to the study of the solvability and uniqueness of the solution to a nonlocal inverse boundary-value problem for a parabolic equation defined on a rectangular domain. Sharp estimates for the solution of a boundary value problem (BVP) involving the determination of a parameter in a parabolic equation, in Hölder norms, are established in [7]. In [15], the authors investigated SIP for a parabolic-type equation with multipoint nonlocal boundary conditions. Namely stability estimates for solution of some SIPs were established. Well-posedness of non-classical BVPs for reverse parabolic PDE with distincts conditions and difference schemes for their numerical solutions was investigated in [14, 16]. The authors in [8] investigated SIP for a parabolic equation with delay. Numerical algorithms for solving source identification problems (SIPs) for telegraph-parabolic and hyperbolic-parabolic equations were developed in [12] and [17]. The uniqueness of a nonlinear inverse problem involving the determination of an orbit function from final-time data measured within a bounded domain is established in [27]. The solvability of certain inverse problems for the nonstationary heat-and-mass transfer system is addressed in [32].

Paper [37] investigates SIP for a one-dimensional heat equation to simultaneously determine the diffusion coefficient, spatially dependent source term, and initial condition using boundary control, measurements, and temperature data at a single time.

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In the paper [35], an inverse problem backward in time was studied and a numerical illustration was presented. Despite extensive research on the well-posedness of SIPs for parabolic differential equations, there are very few scientific results concerning source identification in reverse parabolic equations.

Assume that the operator $\mathbb{A}$ is a self-adjoint positive definite (SAPD) operator in an Hilbert space H and δ is some positive real number such that $\mathbb{A} > \delta I$ for the identity operator I. Suppose that $\Upsilon > 0, \gamma_1, \gamma_2, ..., \gamma_r, \mu_1, \mu_2, ..., \mu_r$ are given numbers and

$$\sum_{k=1}^{r} |\mu_k| < 1, 0 \leq \gamma_1 < \gamma_2 < ... < \gamma_r < \Upsilon \ . \tag{1}$$

Consider SIP to find a pair (p, v) for the following abstract reverse parabolic equation

$$\frac{dv(y)}{dy} - \mathbb{A}\ v(y) = f(y) + p, 0 < y < \Upsilon, \tag{2}$$

with the initial condition

$$v(0) = \xi, \tag{3}$$

and the multi point nonlocal condition

$$v(\Upsilon) = \sum_{k=1}^{r} \mu_k v(\gamma_k) + \zeta. \tag{4}$$

Here H-valued function f and elements ξ, ζ of the Hilbert space H are given.

Let $\beta \in [0, 1]$. In future, we will use symbols $C([0, \Upsilon], H) = C(H)$, $C^{\beta}([0, \Upsilon], H) = C^{\beta}(H)$ and $C_1^{\beta}([0, \Upsilon], H) = C_1^{\beta}(H)$ for notation of the corresponding Banach spaces of funcions $u : [0, \Upsilon] \to H$ with the following norms

$$\|u\|_{C(H)} = \max_{0 \leq y \leq \Upsilon} \|u(y)\|_H, \|u\|_{C^{\beta}(H)} = \|u\|_{C(H)} + \sup_{0 \leq y < y+\tau \leq \Upsilon} \frac{\|u(y+\tau)-u(y)\|_H}{\tau^{\beta}},$$

$$\|u\|_{C_1^{\beta}(H)} = \|u\|_{C(H)} + \sup_{0 \leq t < t+\tau \leq \Upsilon} \frac{(\Upsilon-t)^{\beta}\|u(t+\tau)-u(t)\|_H}{\tau^{\beta}}.$$

Lemma 1.1. *([11], p.16) For any $0 < t < t + \tau \leq \Upsilon$ and $0 \leq \beta \leq 1$ one has the inequalities*

$$\left\|e^{-t\mathbb{A}}\right\|_{H \to H} \leq 1,$$

$$\left\|e^{-t\mathbb{A}} - e^{-(t+\tau)\mathbb{A}}\right\|_{H \to H} \leq \mathcal{M}\frac{\tau^{\beta}}{t^{\beta}}, \tag{5}$$

$$\left\|\mathbb{A}\left(e^{-t\mathbb{A}} - e^{-(t+\tau)\mathbb{A}}\right)\right\|_{H \to H} \leq \mathcal{M}\frac{\tau^{\beta}}{t^{\beta+1}}$$

where $\mathcal{M}$ does not depend on β, τ and t.

Lemma 1.2. *Under conditions (1) there exists an inverse G^{-1} to the operator*

$$G = \ I - \sum_{i=1}^{r} \mu_i \left(I - e^{-\Upsilon\mathbb{A}}\right)^{-1} \left(e^{-(\Upsilon-\gamma_i)\mathbb{A}} - e^{-\Upsilon\mathbb{A}}\right), \tag{6}$$

so that for some positive $\mathcal{M}$

$$\left\|G^{-1}\right\|_{H \to H} \leq \mathcal{M} \tag{7}$$

is valid.

Proof. It is easy to check that

$$\forall\lambda,\forall i\in\{1,2,...,r\}\ \left|\frac{e^{-\Upsilon\lambda}-e^{-(\Upsilon-\gamma_i)\lambda}}{1-e^{-\Upsilon\lambda}}\right|\leq\ \frac{e^{\gamma_i\lambda}-1}{e^{\Upsilon\lambda}-1}\leq 1.$$

Using the spectral resolution theory [26], we get

$$\left\|G^{-1}\right\|_{H\to H}\leq \sup_{\delta\leq\lambda<\infty}\left|1-\sum_{i=1}^{r}\mu_k\tfrac{e^{-\Upsilon\lambda}-e^{-(\Upsilon-\gamma_i)\lambda}}{1-e^{-\Upsilon\lambda}}\right|^{-1}$$

$$\leq \sup_{\delta\leq\lambda<\infty}\left(1-\sum_{i=1}^{r}|\mu_i|\left|\tfrac{e^{-\lambda}-e^{-(\Upsilon-\gamma_i)\lambda}}{1-e^{-\Upsilon\lambda}}\right|\right)^{-1}$$

$$\leq \sup_{\delta\leq\lambda<\infty}\ \left(1-\sum_{i=1}^{r}|\mu_i|\right)^{-1}=\left(1-\sum_{i=1}^{r}|\mu_i|\right)^{-1}\leq\mathcal{M}.$$

Thus, the operator G has the bounded inverse G^{-1}.

The aim of current work is to study SIP for reverse parabolic equation (2)-(4) under the assumptions (1). Existence and uniqueness of the solution are established. Stability estimates for the solution are obtained. Subsequently theoretical results are applied to establish well-posedness of three SIPs for the reverse parabolic PDE. □

2. Stability estimates for abstract SIP

First, we derive a solution formula for SIP (2)-(4). It can be easily checked that (see, [13])

$$v(t)=e^{-(\Upsilon-t)A}v(\Upsilon)-\int_{t}^{\Upsilon}e^{-(\gamma-t)\mathbb{A}}\left(f(\gamma)+p\right)d\gamma \tag{8}$$

satisfies parabolic equation (2). By virtue of (8), we can obtain

$$v(\gamma_k)=e^{-(\Upsilon-\gamma_k)\mathbb{A}}v(\Upsilon)-\int_{\gamma_k}^{\Upsilon}e^{-(\gamma-\gamma_k)\mathbb{A}}f(\gamma)d\gamma-\left(I-e^{-(\Upsilon-\gamma_k)\mathbb{A}}\right)\mathbb{A}^{-1}p \tag{9}$$

for any $k=1,...,r$. Applying formulas (9) to multipoint condition (4) and initial condition (3), we get

$$v(\Upsilon)=\sum_{k=1}^{r}\mu_k e^{-(\Upsilon-\gamma_k)\mathbb{A}}v(1)-\sum_{k=1}^{r}\mu_k\int_{\gamma_k}^{\Upsilon}e^{-(\gamma-\gamma_k)\mathbb{A}}f(s)ds$$

$$-\sum_{k=1}^{r}\mu_k\left(I-e^{-(\Upsilon-\gamma_k)\mathbb{A}}\right)\mathbb{A}^{-1}p+\zeta$$

and

$$v(0)=\xi=e^{-\Upsilon\mathbb{A}}v(\Upsilon)-\int_{0}^{\Upsilon}e^{-\gamma\mathbb{A}}f(\gamma)d\gamma-\int_{0}^{\Upsilon}e^{-\gamma\mathbb{A}}d\gamma p.$$

So, we obtain linear system of equations

$$\left(I-\sum_{k=1}^{r}\mu_k e^{-(1-\gamma_k)\mathbb{A}}\right)v(\Upsilon)+\sum_{k=1}^{\Upsilon}\mu_k\left(I-e^{-(\Upsilon-\gamma_k)\mathbb{A}}\right)\ \mathbb{A}^{-1}p$$

$$=\ \zeta-\sum_{k=1}^{r}\mu_k\int_{\gamma_k}^{\Upsilon}e^{-(s-\gamma_k)\mathbb{A}}f(s)\ ds, \tag{10}$$

$$e^{-\Upsilon A}v(\Upsilon)-\left(I-e^{-\Upsilon\mathbb{A}}\right)\mathbb{A}^{-1}p=\xi+\int_{0}^{\Upsilon}e^{-\gamma\mathbb{A}}f(\gamma)d\gamma$$

to find unknowns $A^{-1}p$ and $v(\Upsilon)$. The system of equations (10) has a determinant operator in the following form

$$D=-\left(I-e^{-\Upsilon A}\right)\left(I-\sum_{k=1}^{r}\mu_k e^{-(\Upsilon-\gamma_k)A}\right)-e^{-A}\sum_{k=1}^{r}\mu_k\left(I-e^{-(\Upsilon-\gamma_k)A}\right).$$

It can be rewritten as

$$D=-\left(I-e^{-\Upsilon\mathbb{A}}\right)\left(I-\sum_{k=1}^{r}\mu_k\left(I-e^{-\Upsilon A}\right)^{-1}\left(e^{-(\Upsilon-\gamma_k)A}-e^{-\Upsilon A}\right)\right).$$

From the last equality and Lemma 1.2 it follows that $v(\Upsilon)$ is defined by

$$v(\Upsilon)=-\left(\zeta-\sum_{k=1}^{r}\mu_k\int_{\gamma_k}^{\Upsilon}e^{-(\gamma-\gamma_k)\mathbb{A}}f(\gamma)d\gamma\right)G^{-1}$$

$$-\left(\xi+\int_{0}^{\Upsilon}e^{-\gamma\mathbb{A}}f(\gamma)d\gamma\right)\ \left(\sum_{k=1}^{r}\mu_k\left(I-e^{-(\Upsilon-\gamma_k)\mathbb{A}}\right)\right)G^{-1}Q$$

and an element $\mathbb{A}^{-1}p$ can be defined by

$$\begin{aligned}\mathbb{A}^{-1}p=&-\left(I-\sum_{k=1}^{r}\mu_k e^{-(\Upsilon-\gamma_k)\mathbb{A}}\right)\left(\xi+\int_{0}^{\Upsilon}e^{-\gamma\mathbb{A}}f(\gamma)d\gamma\right)G^{-1}Q\\&+e^{-\mathbb{A}}\left(\zeta-\sum_{k=1}^{r}\mu_k\int_{\gamma_k}^{\Upsilon}e^{-(\gamma-\gamma_k)\mathbb{A}}f(\gamma)d\gamma\right)G^{-1}Q,\end{aligned} \tag{11}$$

where $Q=\left(I-e^{-\Upsilon A}\right)^{-1}$.

So

$$\begin{aligned}p=&-\left(I-\sum_{k=1}^{r}\mu_k e^{-(\Upsilon-\gamma_k)A}\right)\left(\mathbb{A}\xi+\mathbb{A}\int_{0}^{\Upsilon}e^{-\gamma\mathbb{A}}f(\gamma)d\gamma\right)G^{-1}Q\\&+e^{-\Upsilon A}\left(\mathbb{A}\zeta-\sum_{k=1}^{r}\mu_k\mathbb{A}\int_{\gamma_k}^{\Upsilon}e^{-(\gamma-\gamma_k)\mathbb{A}}f(\gamma)d\gamma\right)G^{-1}Q.\end{aligned} \tag{12}$$

Therefore, SIP (2)-(4) is uniquely solvable. Furthermore, the element p and the function v are defined by the corresponding formulas (12) and (8).

Theorem 2.1. *Assume that (1) are fulfilled, and $\xi\in H,\zeta\in D(\mathbb{A}),f\in C^{\beta}(H)$. Then, for the solution $(p,v(t))$ of SIP (2)-(4) the stability estimates*

$$\|v\|_{C(H)}\leq\mathcal{M}\ \left[\|\xi\|_H+\|\zeta\|_H+\|f\|_{C(H)}\right], \tag{13}$$

$$\|p\|_H \leq \mathcal{M} \left[\|\xi\|_H + \|\mathbb{A}\zeta\|_H + \frac{1}{\beta} \|f\|_{C^\beta(H)} \right] \tag{14}$$

are valid. Here $\mathcal{M}$ *does not depend on* β, ζ, ξ,*and* f.

Proof. (12) can be rewritten as

$$p = L_1 + L_2 + L_3 + L_4,$$

where

$$\begin{aligned}
L_1 &= -\left(I - \sum_{k=1}^{r} \mu_k e^{-(\Upsilon-\gamma_k)\mathbb{A}} \right) \mathbb{A}\xi, \\
L_2 &= -\left(I - \sum_{k=1}^{r} \mu_k e^{-(\Upsilon-\gamma_k)\mathbb{A}} \right) \mathbb{A} \int_0^{\Upsilon} e^{-s\mathbb{A}} f(s)\, ds\, G^{-1}Q, \\
L_3 &= e^{-\Upsilon\mathbb{A}}\mathbb{A}\zeta\, G^{-1}Q, \\
L_4 &= -e^{-T\mathbb{A}} \sum_{k=1}^{r} \mu_k \mathbb{A} \int_{\gamma_k}^{\Upsilon} e^{-(s-\gamma_k)\mathbb{A}} f(s)\, ds\, G^{-1}Q.
\end{aligned}$$

Using (1),(5),(7), Cauchy–Schwarz and triangle inequalities, we can obtain

$$\|L_1\|_H \leq \mathcal{M}_1 \ \|A\xi\|_H, \|L_2\|_H \leq \tfrac{1}{\beta}\mathcal{M}_2 \|f\|_{C^\beta(H)},$$

$$\|L_3\|_H \leq \mathcal{M}_3 \|A\zeta\|_H, \|L_4\|_H \leq \tfrac{1}{\beta}\mathcal{M}_4 \|f\|_{C^\beta(H)}.$$

Combining the last estimates gives (14). Using (8) and (12), we get

$$v(t) = \sum_{k=1}^{6} J_k(t),$$

where

$$\begin{aligned}
J_1 &= J_1(t) = e^{-t\mathbb{A}}\xi, \\
J_2 &= J_2(t) = \int_0^t e^{-(t-s)\mathbb{A}} f(s) ds, \\
J_3 &= Q\zeta, \ J_4 = Q \left(e^{-\Upsilon\mathbb{A}} - \sum_{k=1}^{r} \mu_k e^{-s_k\mathbb{A}} \right) \xi, \\
J_5 &= -\, Q \int_0^{\Upsilon} e^{-(\Upsilon-s)\mathbb{A}} f(s) ds, \\
J_6 &= \ Q \sum_{k=1}^{r} \mu_k \int_0^{s_k} e^{-(s_k-s)\mathbb{A}} f(s) ds.
\end{aligned}$$

From (1), (5), (7), Cauchy–Schwarz and triangle inequalities follow the estimates

$$\|J_1(t)\|_H \leq \mathcal{M} \|\xi\|_H, \|J_2(t)\|_H \leq \mathcal{M}_5 \|f\|_{C(H)},$$

$$\|J_3\|_H \leq \ \mathcal{M}_6 \|\zeta\|_H, \|J_4\|_H \leq \mathcal{M}_7 \|\xi\|_H,$$

$$\|J_5\|_H \leq \ \mathcal{M}_8 \ \|f\|_{C(H)}, \|J_6\|_H \leq \mathcal{M}_9 \|f\|_{C(H)}.$$

Therefore, combining these inequalities, we can establish the validity of estimate (13). □

3. Applications to partial differential equations

Now, we consider an application of abstract problem (2)-(4) to the study of BVPs for the reverse parabolic partial differential equations.

Let $\Omega = (0,l)^n \subset \mathbb{R}^n$ with boundary $S = \partial\Omega, \overline{\Omega} = \Omega \cup S$ and (1) be valid. Moreover, assume that $\xi \in L_2(\Omega), \zeta \in W_2^2(\Omega)$ and $f \in C^{\alpha}(L_2(\Omega))$, a_r are given smooth functions and $a_r(x) \geq a_0 > 0, r = 1, ..., n$, σ is a known positive real number.

First, we study in $[0, \Upsilon] \times \overline{\Omega}$, SIP for a multi-dimensional reverse parabolic PDE with the Dirichlet-type boundary condition

$$\begin{cases} v_y(x,y) + \sum\limits_{i=1}^{n} (a_i(x) v_{x_i}(x,y))_{x_i} - \sigma v(x,y) = f(x,y) + p(x), \\ \\ 0 < y < \Upsilon, x = (x_1, \cdots, x_n) \in \Omega, \\ \\ v(x,0) = \xi(x), \\ \\ v(x,\Upsilon) = \sum\limits_{k=1}^{r} \mu_k v(x, s_k) + \zeta, \ x \in \overline{\Omega}, \\ \\ v(x,y) = 0, \quad 0 \leq y \leq \Upsilon, \ x \in S. \end{cases} \tag{15}$$

To formulate stability results we will use notation $C(\overline{\Omega})$ for the space of continuous functions u defined on $\overline{\Omega}$, equipped with norm $\| u \|_{C(\overline{\Omega})} = \max\limits_{x \in \overline{\Omega}} |u(x)|$ and $L_2(\Omega)$ for the space of square integrable functions u, equipped with norm

$$\| u \|_{L_2(\Omega)} = \left(\int ... \int\limits_{x \in \overline{\Omega}} |u(x)|^2 dx_1 ... dx_n \right)^{\frac{1}{2}}.$$

It is well known that the differential expression

$$\mathbb{A}u(x) = -\sum_{i=1}^{n} (a_r(x) u_{x_i}(x))_{x_i} \tag{16}$$

defines A which is SAPD operator acting on $L_2(\Omega)$ with the domain

$$D(\mathbb{A}) = \{u, u_{x_i x_i} \in L_2(\Omega), \ u(x) = 0 \ \text{ on } S\}.$$

Accordingly, we can replace mixed BVP (15) for a multi-dimensional reverse parabolic equation with the abstract problem (2)-(4). Using the results of Theorem 2.1 we can establish the following statement on the stability of solution (15).

Theorem 3.1. *Assume that the inequalities (1) are valid, $\xi \in L_2(\Omega)$, $\zeta \in W_2^2(\Omega)$, $f \in C^{\beta}(L_2(\Omega))$ are given. Then, for the solution of mixed BVP (15), the following estimates are valid:*

$$\|v\|_{C(L_2(\Omega))} \leq \mathcal{M} \left[\|\xi\|_{L_2(\Omega)} + \|\zeta\|_{L_2(\Omega)} + \|f\|_{C(L_2(\Omega))} \right], \tag{17}$$

$$\|p\|_{L_2(\Omega)} \leq \mathcal{M} \left[\|\xi\|_{L_2(\Omega)} + \|\zeta\|_{W_2^2(\Omega)} + \frac{1}{\beta} \|f\|_{C^{\beta}(L_2(\Omega))} \right], \tag{18}$$

where $\mathcal{M}$ does not depend on β, ζ, ξ, and f.

Second, we study in $[0,\Upsilon]\times\overline{\Omega}$, SIP for a multi-dimensional parabolic PDE with the nonlocal condition and Neumann type boundary condition

$$\begin{cases} v_y(x,y)+\sum\limits_{i=1}^{n}(a_i(x)v_{x_i}(x,y))_{x_i}+\sigma v(x,y)=f(x,y)+p(x), \\ \\ \quad 0<y<\Upsilon, x=(x_1,\cdots,x_n)\in\Omega, \\ v(0,x)=\xi(x), \\ \\ v(x,\Upsilon)=\sum\limits_{k=1}^{r}\mu_k v(\gamma_k,x)+\zeta,\ x\in\overline{\Omega}, \\ \\ \frac{\partial}{\partial n}v(x,y)=0, \quad 0\le y\le\Upsilon,\ x\in S, \end{cases} \tag{19}$$

where n is the normal vector to boundary S.

The differential expression (16) defines $\mathbb{A}$ which is SAPD operator acting on $L_2(\Omega)$ with the corresponding domain

$$D(\mathbb{A})=\left\{u, u_{x_ix_i}\ \in L_2(\Omega),\ \ \frac{\partial}{\partial n}u(x)=0\ \text{ on } S\right\}.$$

Thus, BVP (19) for the multi-dimensional reverse PDE can be reduced to the abstract problem (2)-(4). Using the results of Theorem 2.1 we can establish the following statement on stability of solution (15).

Theorem 3.2. *Suppose that the inequalities (1) are valid, $\xi\in L_2(\Omega)$, $\zeta\in W_2^2(\Omega)$, $f\in C^{\alpha}(L_2(\Omega))$ are given. Then, for the solution of SIP (19), the estimates (17) and (18) are valid, where the positive number $\mathcal{M}$ does not depend on β,ζ,ξ, and f.*

Third, we consider in $[0,l]\times[0,\Upsilon]$, the mixed BVP for a reverse parabolic equation

$$\begin{cases} v_y(x,y)+a(x)\ \ v_{xx}(x,y)-\sigma v(x,y)=f(x,y)+p(x), \\ \\ 0<y\ <\Upsilon, 0<x<l, \\ \\ v(x,\Upsilon)=\sum\limits_{k=1}^{r}\mu_k v(x,\gamma_k)+\zeta,\ v(x,0)=\xi(x), \quad 0\le x\le l, \\ \\ v(0,y)=v(l,y),\ \ v_x(0,y)=v_x(l,y), 0\le y\le\Upsilon. \end{cases} \tag{20}$$

Here $f(t,x)$ and $a(x)$ are given suffuciently smooth functions, a is a positive valued function.

The differential expression

$$\mathbb{A}u(x)=-a(x)\ \ v_{xx}(x,y)+\sigma v(x,y)$$

defines $\mathbb{A}$ which is SAPD operator acting on $L_2(0,l)$ with the domain

$$D(\mathbb{A})=\{u, u_{xx}\in L_2(0,l), \quad u(0)=u(l),\ \ u_x(0)=u_x(l)\}.$$

Thus, we mixed BVP (20) for the parabolic equation can be reduced to the abstract problem (2)-(4). Applying the results of Theorem 2.1, we can establish the following results on the stability of the solution of BVP (20).

Theorem 3.3. *Assume that the inequalities (1) are valid,* $\xi \in L_2(0,l)$, $\zeta \in W_2^2(0,l)$, $f \in C^{\beta}(L_2(0,l))$ *are given. Then, the solution of mixed BVP (15) satisfies the following stability estimates:*

$$\|v\|_{C(L_2(0,l))} \leq \mathcal{M}\left[\|\xi\|_{L_2(0,l)} + \|\zeta\|_{L_2(0,l)} + \|f\|_{C(L_2(0,l))}\right],$$

$$\|p\|_{L_2(0,l)} \leq \mathcal{M}\left[\|\xi\|_{L_2(0,l)} + \|\zeta\|_{W_2^2(0,l)} + \frac{1}{\beta}\|f\|_{C^{\beta}(L_2(0,l))}\right],$$

where positive number $\mathcal{M}$ *does not depend on* β, ζ, ξ, *and* f.

4. Conclusion

In this work, SIP for a reverse parabolic equation with a multi-point nonlocal boundary condition has been studied. Stability estimates for the solution of inverse problem have been obtained. The well-posedness of three SIPs for the reverse parabolic PDSs has been established. In the future, we plan to develop a stable difference scheme for obtaining approximate solutions such type of SIPs.

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