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OFWA OPERATORS AS A COMPUTATIONAL MECHANISM TO AGGREGATE VARIABLES

R.G. ARAGÓN[1,*], J. MEDINA[1], S. MOLINA-RUIZ[1]

Abstract. Modeling scientific and engineering problems requires the aggregation of variables. Ordered Functional Weighted Averaging (OFWA) operators were introduced to provide greater flexibility and versatility on a wide range of scenarios, allowing multiple criteria to be taken into account across different values, than other existing aggregation operators. Although the whole set of OFWA operators forms a complete lattice with the usual pointwise ordering, its complexity does not allow a trivial mechanism to compare them. This paper studies how greater or lesser OFWA operators can be obtained and we shows that significant well-known operators, such as the Choquet integral, are particular cases of the OFWA operator.

Keywords: information aggregation, Ordered Functional Weighted Averaging (OFWA) operator, pointwise order.

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1. Introduction

Information aggregation has become increasingly vital in today's data-driven society for extracting reliable knowledge and supporting informed decisions [15, 17, 25, 26]. Many systems rely on many individual sources to gather local information and operators to obtain a global answer from these variables that is relevant to the environment. However, many classical operators within aggregation theory lack versatility, as they typically apply a single aggregation criterion regardless of the setting [28, 32, 33]. This limitation makes them unsuitable for scenarios where the weights of criteria should adapt dynamically depending on the inputs. In this paper, we focus on Ordered Functional Weighted Averaging (OFWA, for short) operators, introduced in [19], which provide precisely this kind of versatility.

One crucial task within aggregation theory is selecting the most appropriate operator for a specific practical application among the existing operators [1, 13, 14, 20, 27]. A central aspect of this research is the pointwise comparison of operators, particularly within the family of OFWA operators. This comparison naturally leads to computing their supremum and infimum, which are again OFWA operators due to their complete lattice structure [5]. Moreover, the results presented in this paper provide constructive methods for modifying the weights of OFWA operators, thereby defining new operators that yield higher or lower outputs in specified regions. This approach enables the customization of the aggregation process to align with the specific requirements of the user.

A wide variety of aggregation operators have been introduced within information aggregation theory for addressing very different scenarios [8, 31], making the choice far from straightforward. Given the considerable diversity of operators, it is essential to compare any new operator with existing ones, acknowledging their advantages and shortcomings. We will employ a section of

[1]University of Cádiz, Cádiz, Spain
e-mail: jesus.medina@uca.es, samuel.molina@uca.es
[*]Corresponding author: roberto.aragon@uca.es
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this work to carry out precisely this task and compare the notion of OFWA operators to some of the existing families of aggregation operators, showing that some well-known operators are particular cases of OFWA operators.

This paper starts with the necessary preliminaries to understand and motivate the definition of OFWA operators. In Section 3, we introduce criteria to compare two OFWA operators pointwise in a subset of the unit interval. Section 4 introduces a practical example of a photovoltaic facility [3, 4] and how the weights can be modified in order to obtain higher or lower outputs using the results introduced previously. Lastly, in Section 5 we consider different well-known operators within aggregation theory, including the Choquet integral [11], and show that they are particular instances of OFWA operators.

2. Preliminaries

This section will serve as a brief introduction to the notions needed to understand this work, starting with the definition of OWA operators introduced in [32].

Definition 2.1. *Let $W = (w_1, \dots, w_n)$ be a list of weights such that $w_i \in [0,1]$, for all $i \in \{1, \dots, n\}$, and $\sum_{i=1}^{n} w_i = 1$. An (n)-ary ordered weighted averaging (OWA) operator associated with W is a function $F_W : [0,1]^n \to [0,1]$ defined as:*

$$F_W(a_1, \dots, a_n) = \sum_{i=1}^{n} w_i a_{\rho(i)}$$

for all $(a_1, ..., a_n) \in [0,1]^n$, where $\rho : \{1, ..., n\} \to \{1, ..., n\}$ is a permutation on the index set satisfying that $a_{\rho(n)} \leq ... \leq a_{\rho(1)}$.

This class of operators possesses several important properties. For instance, they are isotone operators and satisfy some boundary conditions, i.e., they are aggregation operators. OWA operators have also proven useful in many practical applications [9, 18, 30, 32]. However, these operators use only one criterion to aggregate all the variables, which can be undesirable in certain situations, as next example shows.

Example 2.1. Consider a series of temperature sensors that need to be aggregated in order to obtain a global value for the system. Some experimental testing shows that these sensors are more reliable in higher temperatures, so we may give higher weights to those outputting higher values. Assume, for instance, that the values are aggregated using the OWA operator defined by the weights

$$W = \left(\frac{3}{12}, \frac{3}{12}, \frac{2}{12}, \frac{2}{12}, \frac{1}{12}, \frac{1}{12} \right).$$

Now, take the following two lists of values to be aggregated:

$$a = (0.8, 0.7, 0.7, 0.7, 0.6, 0.5),$$
$$b = (0.8, 0.3, 0.3, 0.3, 0.2, 0.2).$$

We can clearly see that the values of (a) are all around 0.7, with some variation on the data, so the aggregated value should lie near 0.7. Indeed,

$$F_W(a) = \sum_{i=1}^{6} w_i a_{\rho(i)} = 0.7.$$

However, in the second list, the values all lies between 0.2 and 0.3 except the greatest, which differs greatly from the rest. The aggregated value using the OWA operator is

$$F_W(b) = \sum_{i=1}^{6} w_i b_{\rho(i)} \approx 0.4.$$

The heavy weight on the greatest value makes the resulting output higher than expected. In this scenario, using the same criteria via the weights (W) is undesirable since the input values may indicate signs of malfunction. In this scenario, a more appropriate list of weights for the aggregation is

$$W' = \left(0, \frac{3}{9}, \frac{2}{9}, \frac{2}{9}, \frac{1}{9}, \frac{1}{9}\right).$$

The resulting output using the OWA operator associated with this list of weights is

$$F_{W'}(b) = \sum_{i=1}^{6} w'_i b_{\rho(i)} \approx 0.28$$

which aligns better with our expectations.

OWA operators are defined using only one list of weights, aggregating the variables according to one criterion. Certain scenarios require multiple criteria to be considered depending on the input values. In order to solve this, OFWA operators were introduced in [19] as a generalization of OWA operators by allowing the weights to change.

Definition 2.2. *Let $\Omega = (\omega_1, \dots, \omega_n)$ be a list of functional weights such that $\omega_i \colon [0,1]^n \to [0,1]$ and $\sum_{i=1}^{n} \omega_i(a_1, \dots, a_n) = 1$, for all $(a_1, \dots, a_n) \in [0,1]^n$. An n-ary ordered functional weighted averaging (OFWA) operator associated with Ω is a function $F_\Omega : [0,1]^n \to [0,1]$ defined as:*

$$F_\Omega(a_1, \dots, a_n) = \sum_{i=1}^{n} \omega_i(a_1, \dots, a_n) a_{\rho(i)},$$

where $\rho \colon \{1, \dots, n\} \to \{1, \dots, n\}$ is a permutation on the index set satisfying that $a_{\rho(n)} \leq \cdots \leq a_{\rho(1)}$.

OFWA operators also satisfy the same boundary conditions as OWA operators but they are not necessarily increasing. Sufficient conditions for ensuring monotonicity were introduced in [19]. The next example will introduce a particular class of OFWA operators based on the exponential function.

Example 2.2. Given $k, s_1, \dots, s_n \in \mathbb{R}^+$ positive numbers and $m_1, \dots, m_n \in [0,1]$ such that $m_n \leq \cdots \leq m_1$, we can define the weights $\omega_i \colon [0,1]^n \to [0,1]$ as

$$\omega_i(a_1, \dots, a_n) = \frac{1 - m_i\, e^{-k(a_1)^{s_1} \cdots (a_n)^{s_n}}}{\Gamma(a_1, \dots, a_n)},$$

where $\Gamma(a_1, \dots, a_n) = n - (m_1 + \cdots + m_n)\, e^{-k(a_1)^{s_1} \cdots (a_n)^{s_n}}$, for all $(a_1, \dots, a_n) \in [0,1]^n$. The associated OFWA operator $F_{\Omega_\gamma} \colon [0,1]^n \to [0,1]$, with $\Omega_\gamma = (\omega_1, \dots, \omega_n)$, is increasing, see [19, Theorem 28] for more details.

In addition to satisfying the boundary conditions, OFWA operators are also bounded, that is, they lie between the minimum and maximum operator. Recently, the authors of [12] proved the converse, that every bounded operator can be represented as an OFWA operator.

Theorem 2.1. *Let $f : [0,1]^n \to [0,1]$ be an n-ary operator satisfying that*

$$\min\{a_1, \dots, a_n\} \leq f(a_1, \dots, a_n) \leq \max\{a_1, \dots, a_n\}$$

for all $(a_1, \dots, a_n) \in [0,1]^n$. Then, there is a list of weights $\Omega_f = (\omega_1^f, \dots, \omega_n^f)$ such that $F_{\Omega_f} = f$.

A particular consequence to the theorem above is that every idempotent aggregation function is an OFWA. Indeed, if $f : [0,1]^n \to [0,1]$ is idempotent and increasing, then

$$a_{\rho(n)} = f(a_{\rho(n)}, \dots, a_{\rho(n)}) \leq f(a_1, \dots, a_n) \leq f(a_{\rho(1)}, \dots, a_{\rho(1)}) = a_{\rho(1)}$$

for all $(a_1, \dots, a_n) \in [0,1]^n$.

3. Comparing OFWA operators

Many scenarios require users to select an aggregation operator from a given family of options, either to determine the most suitable one for the situation or to construct a new one based on specific conditions. This naturally leads to the question of which aggregation operators can be compared. In this section, we introduce three sufficient conditions on the weights that establish when an OFWA operator is greater than or equal to another. All the results presented will apply to a subset of the Cartesian Product $[0,1]^n$, which can be interpreted pointwise or across the entire domain, so we will use the following notation. Given a mapping $f : [0,1]^n \to [0,1]$ and a subset $A \subseteq [0,1]^n$, the restriction of f to A is the mapping $f_{|A}\colon A \to [0,1]$ defined as $f_{|A}(\mathbf{a}) = f(\mathbf{a})$, for all $\mathbf{a} \in A$. The first result is stated in the following proposition and it compares the OFWA operators by the sum of their weights.

Proposition 3.1. *Given two lists of functional weights $\Omega = (\omega_1, \dots, \omega_n)$ and $\Omega' = (\omega'_1, \dots, \omega'_n)$, the associated OFWA operators F_Ω and $F_{\Omega'}$ and a subset $A \subseteq [0,1]^n$, if the weights satisfy that*

$$\sum_{i=1}^{m} \omega_i(a_1, \dots, a_n) \leq \sum_{i=1}^{m} \omega'_i(a_1, \dots, a_n)$$

for all $m \in \{1, \dots, n-1\}$ and $(a_1, \dots, a_n) \in A$, then $(F_\Omega)_{|A} \leq (F_{\Omega'})_{|A}$.

Proof. We will consider an element $a_1, ..., a_n \in [0,1]^n$ and we will denote $\alpha_i = \omega_i(a_1, \dots, a_n)$ and $\alpha'_i = \omega'_i(a_1, \dots, a_n)$. Moreover, to simplify the notation with respect to the permutations, we will assume, without loss of generality, that $a_{i+1} \leq a_i$, for all $i \in \{1, \dots, n-1\}$. First of all, given $m \in \{1, \dots, n-1\}$, we have that

$$\begin{aligned}(\sum_{i=1}^{m} \alpha'_i - \sum_{i=1}^{m} \alpha_i)a_{m+1} &\leq (\sum_{i=1}^{m} \alpha'_i - \sum_{i=1}^{m} \alpha_i)a_m \\ &= a_m\alpha'_m - a_m\alpha_m + (\sum_{i=1}^{m-1} \alpha'_i - \sum_{i=1}^{m-1} \alpha_i)a_m, \end{aligned} \tag{1}$$

where in the inequality we have used the hypothesis and that $a_{m+1} \leq a_m$. Note that, when $m = 1$, the last term in the expression is just $a_1\alpha'_1 - a_1\alpha_1$.

Now, we will focus on proving the inequality $F_\Omega(a_1, \dots, a_n) \leq F_{\Omega'}(a_1, \dots, a_n)$:

$$\begin{aligned}&\sum_{i=1}^{n} \alpha_i a_i = \alpha_1 a_1 + \cdots + \alpha_{n-1}a_{n-1} + \alpha_n a_n \\ &\overset{(*)}{=} \alpha_1 a_1 + \cdots + \alpha_{n-1}a_{n-1} + \big(\alpha'_n + \sum_{i=1}^{n-1} \alpha'_i - \sum_{i=1}^{n-1} \alpha_i\big)a_n \\ &\overset{(**)}{\leq} \alpha_1 a_1 + \cdots + \alpha_{n-1}a_{n-1} + \alpha'_{n-1}a_{n-1} - \alpha_{n-1}a_{n-1} \\ &+ (\sum_{i=1}^{n-2} \alpha'_i - \sum_{i=1}^{n-2} \alpha_i)a_{n-1} + \alpha'_n a_n \\ &= \alpha_1 a_1 + \cdots + \alpha_{n-2}a_{n-2} + (\sum_{i=1}^{n-2} \alpha'_i - \sum_{i=1}^{n-2} \alpha_i)a_{n-1} + \alpha'_{n-1}a_{n-1} + \alpha'_n a_n,\end{aligned}$$

where $(*)$ is given by the equalities $\sum_{i=1}^{n} \alpha'_i = 1 = \sum_{i=1}^{n} \alpha_i$, and $(**)$ holds by Eqn. (1) and reordering the terms. Therefore, applying sequentially Eqn. (1) we finally obtain the inequality:

$$\alpha_1 a_1 + \cdots + \alpha_{n-1}a_{n-1} + \alpha_n a_n \leq \alpha'_1 a_1 + \cdots + \alpha'_{n-1}a_{n-1} + \alpha'_n a_n.$$

□

The previous result turns into a characterization when dealing with OWA operators:

Proposition 3.2. *Given two lists of weights $W = (w_1, \dots, w_n)$ and $W' = (w'_1, \dots, w'_n)$ and the associated OWA operators F_W and $F_{W'}$, then the weights satisfy*

$$\sum_{i=1}^{m} w_i \leq \sum_{i=1}^{m} w'_i$$

for all $m \in \{1, \dots, n-1\}$ if and only if $F_W \leq F_{W'}$.

Proof. The sufficient condition is proven by the same result for OFWA operators (Proposition 3.1). Now, assume $F_W \leq F_{W'}$ and let $m \in \{1, \dots, n-1\}$. If we define $\mathbf{a}_m = (a_{m1}, \dots, a_{mn})$ as

$$a_{mi} = \begin{cases} 1 & if \ \ (i \leq m) \\ 0 & if \ \ (m < i) \end{cases}$$

for all $i \in \{1, \dots, n\}$, then

$$F_W(\mathbf{a}_m) = w_1 + \cdots + w_m \leq F_{W'}(\mathbf{a}_m) = w'_1 + \cdots + w'_m.$$

□

The previous result suggests that we can compare OWA operators by only checking their values at $n-1$ points, namely the points $(\mathbf{a}_1, \dots, \mathbf{a}_n)$ defined in the proof above.

A simple way to verify the condition of the previous result is comparing the weights one by one, as stated in the following proposition.

Proposition 3.3. *Given two lists of functional weights $\Omega = (\omega_1, \dots, \omega_n)$ and $\Omega' = (\omega'_1, \dots, \omega'_n)$, the associated OFWA operators F_Ω and $F_{\Omega'}$ and a subset $A \subseteq [0,1]^n$, if for every $\mathbf{a} = a_1, \dots, a_n) \in A$ there exist a $k_\mathbf{a} \in \{1, \dots, n-1\}$ such that*

- $\omega_i(a_1, \dots, a_n) \leq \omega'_i(a_1, \dots, a_n), \ \ for \ all \ \ i \in 1, \dots, k_a$
- $\omega_i(a_1, \dots, a_n) \geq \omega'_i(a_1, \dots, a_n), \ \ for \ all \ \ i \in k_a + 1, \dots, n$

then $(F_\Omega)_{|A} \leq (F_{\Omega'})_{|A}$.

Proof. Let $\mathbf{a} = (a_1, \dots, a_n) \in A$ and the corresponding index $k_\mathbf{a} \in \{1, \dots, n-1\}$. We will show that, for every $m \in \{1, \dots, n-1\}$, the OFWA operators satisfy the hypothesis of Proposition 3.1.

If $m \in \{1, \dots, k_\mathbf{a}\}$, the hypothesis state that $\omega_i(a_1, \dots, a_n) \leq \omega'_i(a_1, \dots, a_n)$, for all $i \in \{1, ..., m\}$. Therefore,

$$\sum_{i=1}^{m} \omega_i(a_1, \dots, a_n) \leq \sum_{i=1}^{m} \omega'_i(a_1, \dots, a_n).$$

However, if $m \in \{k_\mathbf{a} + 1, \dots, n-1\}$, then we have that $\omega_i(a_1, \dots, a_n) \geq \omega'_i(a_1, \dots, a_n)$, for all $i \in \{m, \dots, n-1\}$, by the hypothesis, and thus

$$\sum_{i=m}^{n} \omega_i(a_1, \dots, a_n) \geq \sum_{i=m}^{n} \omega'_i(a_1, \dots, a_n).$$

Since $\sum_{i=1}^{n} \omega_i(a_1, \ldots, a_n) = 1 = \sum_{i=1}^{n} \omega'_i(a_1, \ldots, a_n)$, we obtain that

$$\begin{aligned}
\sum_{i=1}^{m} \omega_i(a_1, \ldots, a_n) &= \sum_{i=1}^{n} \omega_i(a_1, \ldots, a_n) - \sum_{i=m+1}^{n} \omega_i(a_1, \ldots, a_n) \\
&= \sum_{i=1}^{n} \omega'_i(a_1, \ldots, a_n) - \sum_{i=m+1}^{n} \omega_i(a_1, \ldots, a_n) \\
&\leq \sum_{i=1}^{n} \omega'_i(a_1, \ldots, a_n) - \sum_{i=m+1}^{n} \omega'_i(a_1, \ldots, a_n) \\
&= \sum_{i=1}^{m} \omega'_i(a_1, \ldots, a_n).
\end{aligned}$$

Therefore, we are under the hypothesis of Proposition 3.1, since the choice of $\mathbf{a} \in A$ was arbitrary. Consequently, we can conclude that $(F_\Omega)_{|A} \leq (F_{\Omega'})_{|A}$. □

Remark 3.1. The previous result can be used to show that in dimension (2), comparing OFWA is equivalent to comparing their first weight: Given $\Omega = (\omega_1, \omega_2)$ and $\Omega' = (\omega'_1, \omega'_2)$, if $\omega_1 \leq \omega'_1$ then $\omega_2 = 1 - \omega_1 \geq 1 - \omega'_1 = \omega'_2$, and via Proposition 3.3 we conclude that $F_\Omega \leq F_{\Omega'}$. Moreover, whenever the weights are constant 2-ary OWA operators, the greater operator will be whichever has the greater first weight. Therefore, the set of 2-ary OWA operators are all comparable, that is, it is a chain where the top element is the maximum operator and the bottom element is the minimum operator. This fact also appears in [5].

The proof of Proposition 3.3 uses Proposition 3.1, thus the latter implies the former. Regarding 2-ary OFWA operators, the two conditions are equivalent.

Corollary 3.1. *Let F_Ω and $F_{\Omega'}$ be two 2-ary OFWA operators and $A \subseteq [0,1]^2$ be a subset. The operators satisfy the condition of Proposition 3.1 in A if and only if they satisfy the condition of Proposition 3.3 in A.*

However, when dealing with n-ary OFWA operators, with $n > 2$, there is no equivalence, as the next example illustrates.

Example 3.1. Let $W = (0.6, 0.1, 0.3)$, $W' = (0.7, 0.3, 0.0)$ be two lists of weights and F_W, $F_{W'}$ the associated OWA operators. It is easy to check that the operators satisfy the condition of Proposition 3.1, thus we have $F_W \leq F_{W'}$, but they do not satisfy the condition of Proposition 3.3. This example can be adjusted to show that the conditions are not equivalent for any arity greater than 2.

In [19, Theorem 22], the authors state a sufficient condition for the monotonicity of OFWA operators based on ratio-increasing operators. A similar condition can be stated to compare OFWA operators, which we will define as follows.

Definition 3.1. *Given two lists of functional weights $\Omega = (\omega_1, \ldots, \omega_n)$ and $\Omega' = (\omega'_1, \ldots, \omega'_n)$, the associated OFWA operators F_Ω and $F_{\Omega'}$ and a subset $A \subseteq [0,1]^n$, we say that $F_{\Omega'}$ is in ratios greater than F_Ω in A if*

$$\omega_i(a_1, \ldots, a_n)\omega'_{i+1}(a_1, \ldots, a_n) \leq \omega'_i(a_1, \ldots, a_n)\omega_{i+1}(a_1, \ldots, a_n)$$

for all $i \in \{1, \ldots, n-1\}$ and $(a_1, \ldots, a_n) \in A$.

The next result shows that the previous condition is sufficient to guarantee that $F_{\Omega'}$ is greater than F_Ω.

Theorem 3.1. *Given two lists of functional weights $\Omega = (\omega_1, \dots, \omega_n)$ and $\Omega' = (\omega'_1, \dots, \omega'_n)$, the associated OFWA operators F_Ω and $F_{\Omega'}$ and a subset $A \subseteq [0,1]^n$, if $F_{\Omega'}$ is in ratios greater than F_Ω in A, then $(F_\Omega)_{|A} \leq (F_{\Omega'})_{|A}$.*

Proof. Given $\mathbf{a} = (a_1, \dots, a_n) \in A$, define $\alpha_i = \omega_i(a_1, \dots, a_n)$ and $\alpha'_i = \omega'_i(a_1, \dots, a_n)$, for all $i \in \{1, \dots, n\}$. We will start assuming that $\alpha_i \leq \alpha'_i$, for all $i \in \{1, \dots, n\}$. Since the weights all add up to 1, we have that

$$1 = \sum_{i=1}^{n} \alpha_i \leq \sum_{i=1}^{n} \alpha'_i = 1.$$

Thus, α_i must be equal to α'_i, for all $i \in \{1, \dots, n\}$, and so $(F_\Omega)_{|A} \leq (F_{\Omega'})_{|A}$ trivially.

If $\alpha_1 \geq \alpha'_1$, since $F_{\Omega'}$ is in ratios greater than F_Ω,

$$\alpha_1 \alpha'_2 \leq \alpha_2 \alpha'_1.$$

Therefore, we must have $\alpha_2 \geq \alpha'_2$. Repeating sequentially this process, we obtain $\alpha_i \geq \alpha'_i$, for all $i \in \{1, \dots, n\}$. Following a similar argument to before, we obtain $\alpha_i = \alpha'_i$, for all $i \in \{1, \dots, n\}$, and we have $(F_\Omega)_{|A} \leq (F_{\Omega'})_{|A}$.

Otherwise, there exists $k_{\mathbf{a}} \in \{1, \dots, n-1\}$, such that $\alpha_i \leq \alpha'_i$, for all $i \in \{1, \dots, k_{\mathbf{a}}\}$, and $\alpha_{k_{\mathbf{a}}+1} \geq \alpha'_{k_{\mathbf{a}}+1}$. Furthermore, since $F_{\Omega'}$ is in ratios greater than F_Ω, we obtain that

$$\alpha_{k_{\mathbf{a}}+1} \alpha'_{k_{\mathbf{a}}+2} \leq \alpha_{k_{\mathbf{a}}+2} \alpha'_{k_{\mathbf{a}}+1}$$

Therefore, we must have $\alpha_{k_{\mathbf{a}}+2} \geq \alpha'_{k_{\mathbf{a}}+2}$. Applying sequentially this process, we obtain $\alpha_i \geq \alpha'_i$, for all $i \in \{k_{\mathbf{a}} + 1, \dots, n\}$. Hence, we are under the assumptions of Proposition 3.3 and we conclude that $(F_\Omega)_{|A} \leq (F_{\Omega'})_{|A}$. □

The condition stated in Theorem 3.1 implies the one in Proposition 3.3, since this one is used in its proof. However, in dimension 2 they are equivalent. If $\omega_1 \leq \omega'_1$, then $\omega_2 = 1 - \omega_1 \geq 1 - \omega'_1 = \omega'_2$ and thus

$$\omega_1 \omega'_2 \leq \omega_2 \omega'_1.$$

This does not happen in general, as the next example illustrates.

Example 3.2. We will consider the simplest case of OFWA operator, where the weights are constant, that is, an OWA operator. Let $W = (0.6, 0.1, 0.3)$, $W' = (0.7, 0.3, 0.0)$ be the two lists of constant weights of Example 3.1, and F_W, $F_{W'}$ be their associated OWA operators. The weights satisfy the conditions of Proposition 3.3 for $k = 2$ since

$$0.6 \leq 0.7 \quad 0.1 \leq 0.3 \quad 0.3 \geq 0.0.$$

Thus, $F_W \leq F_{W'}$. However, $F_{W'}$ is in ratio not greater than F_W since

$$0.6 \cdot 0.3 = 0.18 > 0.1 \cdot 0.7 = 0.07$$

Even though both Proposition 3.1 and Proposition 3.3 are more general results than Theorem 3.1, this one is the easiest to verify in one common scenario. Some OFWA operators, such as the ones presented in Example 3.2, are defined by normalizing positive functions: Fiven a list of functions $\Theta = (\theta_1, \dots, \theta_n)$, where $\theta_i \colon [0,1]^n \to (0, \infty)$, for all $i \in \{1, \dots, n\}$, we can define the list of weights $\Omega = (\omega_1, \dots, \omega_n)$ as

$$\omega_i(a_1, \dots, a_n) = \frac{\theta_i(a_1, \dots, a_n)}{\sum_{j=1}^{n} \theta_j(a_1, \dots, a_n)}$$

for all $i \in \{1, \dots, n\}$ and $(a_1, \dots, a_n) \in [0,1]^n$. These weights trivially add up to 1 and thus F_Ω is an OFWA operator. Given two lists of such positive functions, $\Theta = (\theta_1, \dots, \theta_n)$ and $\Theta' = (\theta'_1, \dots, \theta'_n)$, it can result challenging to compare their associated weights, $\Omega = (\omega_1, \dots, \omega_n)$

and $\Omega' = (\omega'_1, \ldots, \omega'_n)$, or their sums to invoke Proposition 3.1 or Proposition 3.3. On the other hand, the condition in Definition 3.1 in this instance is

$$\frac{\theta_i(\mathbf{a})}{\sum_{j=1}^{n}\theta_j(\mathbf{a})}\cdot\frac{\theta'_{i+1}(\mathbf{a})}{\sum_{j=1}^{n}\theta'_j(\mathbf{a})} \leq \frac{\theta'_i(\mathbf{a})}{\sum_{j=1}^{n}\theta'_j(\mathbf{a})}\cdot\frac{\theta_{i+1}(\mathbf{a})}{\sum_{j=1}^{n}\theta_j(\mathbf{a})}$$

for all $\mathbf{a} \in [0,1]^n$. Since the denominators are the same, the condition can be simplified to

$$\theta_i(\mathbf{a}) \cdot \theta'_{i+1}(\mathbf{a}) \leq \theta'_i(\mathbf{a}) \cdot \theta_{i+1}(\mathbf{a}).$$

Therefore, in this scenario, it is more straightforward to verify the conditions of Theorem 3.1 than the previous ones. The next result illustrates this fact for the OFWA operators from Example 2.2.

Proposition 3.4. *Let $k, k', s, r, s', r' \in \mathbb{R}^+$ be positive numbers and $m_1, m_2, m'_1, m'_2 \in [0,1]$ such that*

(1) $m_2 \leq m_1 \quad$ *and* $\quad m'_2 \leq m'_1$.
(2) $m_1 m'_2 \leq m_2 m'_1$.
(3) $m_1 - m_2 \geq m'_1 - m'_2$.
(4) $k \leq k', \;\; s' \leq s \;$ *and* $\; r' \leq r$.

The OFWA operators $F_\Omega, F_{\Omega'} : [0,1]^2 \to [0,1]$ defined by the weights

$$\omega_1(a_1,a_2) = \frac{1 - m_1\, e^{-k(a_1)^s(a_2)^r}}{\Gamma(a_1,a_2)} \qquad \omega_2(a_1,a_2) = \frac{1 - m_2\, e^{-k(a_1)^s(a_2)^r}}{\Gamma(a_1,a_2)}$$

$$\omega'_1(a_1,a_2) = \frac{1 - m'_1\, e^{-k'(a_1)^{s'}(a_2)^{r'}}}{\Gamma'(a_1,a_2)} \qquad \omega'_2(a_1,a_2) = \frac{1 - m'_2\, e^{-k'(a_1)^{s'}(a_2)^{r'}}}{\Gamma'(a_1,a_2)},$$

where $\Gamma(a_1,a_2) = 2 - (m_1 + m_2)\, e^{-k(a_1)^s(a_2)^r}$ and $\Gamma'(a_1,a_2) = 2 - (m'_1 + m'_2) e^{-k'(a_1)^{s'}(a_2)^{r'}}$, for all $(a_1,a_2) \in [0,1]^2$, satisfy that $F_\Omega \leq F_{\Omega'}$ in $[0,1]^2$.

Proof. We will prove that $F_\Omega \leq F_{\Omega'}$ by checking that $F_{\Omega'}$ is in ratios greater than F_Ω in all $[0,1]^2$, that is,

$$\omega_1(a_1,a_2)\omega_2(a_1,a_2) \leq \omega_2(a_1,a_2)\omega'_1(a_1,a_2).$$

Let $(a_1,a_2) \in [0,1]^2$. Since the denominator in both terms is $\Gamma(a_1,a_2) \cdot \Gamma'(a_1,a_2)$, the previous expression is equivalent to

$$\left(1 - m_1\, e^{-k(a_1)^s(a_2)^r}\right)\left(1 - m'_2\, e^{-k'(a_1)^{s'}(a_2)^{r'}}\right) \leq \left(1 - m_2\, e^{-k(a_1)^s(a_2)^r}\right)\left(1 - m'_1\, e^{-k'(a_1)^{s'}(a_2)^{r'}}\right).$$

Some simple calculations lead us to the following:

$$\begin{aligned}&(m_1 - m_2)\, e^{-k(a_1)^s(a_2)^r} - (m'_1 - m'_2)\, e^{-k'(a_1)^{s'}(a_2)^{r'}} \\ &+ (m_2 m'_1 - m_1 m'_2)\, e^{-k(a_1)^s(a_2)^r}\, e^{-k'(a_1)^{s'}(a_2)^{r'}} \geq 0. \end{aligned} \tag{2}$$

We will divide the previous expression in two terms. The second one,

$$(m_2 m'_1 - m_1 m'_2) e^{-k(a_1)^s(a_2)^r} e^{-k'(a_1)^{s'}(a_2)^{r'}},$$

is positive since the exponential function is positive and hypothesis 2. As for the first one, hypothesis 4 implies that $e^{-k(a_1)^s(a_2)^r} \geq e^{-k'(a_1)^{s'}(a_2)^{r'}}$. Therefore, using hypothesis 1, we obtain

$$\begin{aligned}&(m_1 - m_2)\, e^{-k(a_1)^s(a_2)^r} - (m'_1 - m'_2)\, e^{-k'(a_1)^{s'}(a_2)^{r'}} \geq (m_1 - m_2)\, e^{-k'(a_1)^{s'}(a_2)^{r'}} \\ &- (m'_1 - m'_2)\, e^{-k'(a_1)^{s'}(a_2)^{r'}} = \left(m_1 - m_2 - m'_1 + m'_2\right) e^{-k'(a_1)^{s'}(a_2)^{r'}}.\end{aligned}$$

Applying hypothesis 3 and noting that the exponential function is positive, this term is also positive and hence Inequality (2) is true. Since $F_{\Omega'}$ is in ratios greater than F_Ω, Theorem 3.1 implies that $F_\Omega \leq F_{\Omega'}$ in $[0,1]^2$. □

In the following, we provide an illustrative example of Proposition 3.4.

Example 3.3. Consider the OFWA operators $F_\Omega, F_{\Omega'} : [0,1]^2 \to [0,1]$ defined by the weights

$$\omega_1(a_1,a_2) = \frac{1-0.9\ e^{-5(a_1)^4(a_2)^{10}}}{2-1.1\ e^{-5(a_1)^4(a_2)^{10}}} \qquad \omega_2(a_1,a_2) = \frac{1-0.2\ e^{-5(a_1)^4(a_2)^{10}}}{2-1.1\ e^{-5(a_1)^4(a_2)^{10}}},$$

$$\omega'_1(a_1,a_2) = \frac{1-0.3\ e^{-10(a_1)^3(a_2)^1}}{2-0.3\ e^{-10(a_1)^3(a_2)^1}} \qquad \omega'_2(a_1,a_2) = \frac{1}{2-0.3\ e^{-10(a_1)^3(a_2)^1}}.$$

We are under the hypotheses of Proposition 3.4 with

$$m_1 = 0.9, \quad m_2 = 0.2, \quad k = 5, \quad s = 4, \quad r = 10,$$
$$m'_1 = 0.3, \quad m'_2 = 0.0, \quad k' = 10, \quad s' = 3, \quad r' = 1,$$

since $0 = m_1 m'_2 \leq m_2 m'_1 = 0.06$, $0.7 = m_1 - m_2 \geq m'_1 - m'_2 = 0.3$, $k \leq k'$, $s' \leq s$ and $r' \leq r$. Therefore, we conclude that $F_\Omega \leq F_{\Omega'}$ in $[0,1]^2$. Indeed, Figure 1, where we represent the OFWA operators from two different perspectives, clearly shows this fact.

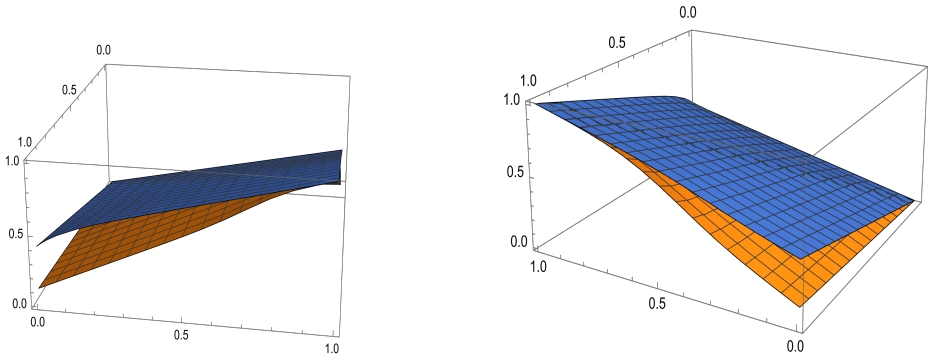


Figure 1. The graph of F_Ω (orange) together with the graph of $F_{\Omega'}$ (blue) in two different perspectives.

Furthermore, Proposition 3.4 can be generalized to arbitrary dimension, as the next result states.

Proposition 3.5. *Let $k, k', s_1, \ldots, s_n, s'_1, \ldots, s'_n \in \mathbb{R}^+$ be positive numbers and $m_1, \ldots, m_n, m'_1, \ldots, m'_n \in [0,1]$ such that*

(1) $m_i \geq m_{i+1}$ *and* $m'_i \geq m'_{i+1}$, *for all* $i \in \{1, \ldots, n-1\}$.
(2) $m_i m'_{i+1} \leq m_{i+1} m'_i$, *for all* $i \in \{1, \ldots, n-1\}$.
(3) $m_i - m_{i+1} \geq m'_i - m'_{i+1}$, *for all* $i \in \{1, \ldots, n-1\}$.
(4) $k \leq k'$, *and* $s'_i \leq s_i$, *for all* $i \in \{1, \ldots, n\}$.

The OFWA operators $F_\Omega, F_{\Omega'} \colon [0,1]^n \to [0,1]$ defined by the weights

$$\omega_i(a_1, \ldots, a_n) = \frac{1 - m_i\ e^{-k(a_1)^{s_1} \cdots (a_n)^{s_n}}}{\Gamma(a_1, \ldots, a_n)},$$

$$\omega'_i(a_1, \ldots, a_n) = \frac{1 - m'_i\ e^{-k'(a_1)^{s'_1} \cdots (a_n)^{s'_n}}}{\Gamma'(a_1, \ldots, a_n)},$$

where $\Gamma(a_1, \ldots, a_n) = n - (m_1 + \cdots + m_n)\ e^{-k(a_1)^{s_1} \cdots (a_n)^{s_n}})$ and $\Gamma'(a_1, \ldots, a_n) = n - (m'_1 + \cdots + m'_n)\ e^{-k'(a_1)^{s'_1} \cdots (a_n)^{s'_n}}$, for all $i \in \{1, \ldots, n\}$ and $(a_1, \ldots, a_n) \in [0,1]^n$, satisfy that

$$F_\Omega \leq F_{\Omega'} \ \text{ in } \ [0,1]^n.$$

Proof. The proof follows analogously to Proposition 3.4.

□

4. A practical example

This section will introduce an example illustrating how OFWA operators work and how the different results of this paper can be used in a practical environment to modify an OFWA operator according to external information. The example is based on the problem considered in [4] and follows the one presented in [5].

A photovoltaic installation consists of 7 inverters I_1 through I_7, each generating a value within $[0, 1]$ that represents its performance at any given moment. To obtain the overall output of the installation, the individual values are aggregated, making use of an OFWA operators in the following way. The two highest and two lowest values are discarded and the average of the remaining three is calculated. However, the inverters I_6 and I_7 are very reliable when they output low values. Therefore, whenever both their outputs are below 0.4, we do not exclude the lower values and average the remaining. Moreover, the inverters I_1 and I_2 are accurate with high values. Consequently, the higher values will not be omitted when their outputs are above 0.7. The resulting operator will be the OFWA operator F_Ω associated with the list of weights $\Omega = (\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7)$ defined as

$$\omega_1(a) = \omega_2(a) = \begin{cases} 1/7 & \text{if } a \in A_1 \\ 1/5 & \text{if } a \in A_2 \\ 0 & \text{if } a \in A_3 \\ 0 & \text{if} a \in A_4 \end{cases}$$

$$\omega_3(a) = \omega_4(a) = \omega_5(a) = \begin{cases} 1/7 & \text{if } a \in A_1 \\ 1/5 & \text{if} a \in A_2 \\ 1/5 & \text{if} a \in A_3 \\ 1/3 & \text{if } a \in A_4 \end{cases}$$

$$\omega_6(a) = \omega_7(a) = \begin{cases} 1/7 & \text{if} a \in A_1 \\ 0 & \text{if } a \in A_2 \\ 1/5 & \text{if} a \in A_3 \\ 0 & \text{if} a \in A_4 \end{cases}$$

$$A_1 = [0.7, 1]^2 \times [0, 1]^3 \times [0, 0.4]^2 \qquad A_2 = [0.7, 1]^2 \times [0, 1]^3 \times (0.4, 1]^2$$
$$A_3 = [0, 0.7)^2 \times [0, 1]^3 \times [0, 0.4]^2 \qquad A_4 = [0, 0.7)^2 \times [0, 1]^3 \times (0.4, 1]^2$$

$$\omega_1'(a) = \omega_2'(a) = \begin{cases} 1/7 & \text{if } a \in A_1 \\ 1/5 & \text{if } a \in A_2 \\ 0 & \text{if } a \in A_3 \\ 0 & \text{if } a \in A_4 \end{cases} \qquad \omega_3'(a) = \omega_4'(a) = \begin{cases} 1/7 & \text{if } a \in A_1 \\ 1/5 & \text{if } a \in A_2 \\ 0.25 & \text{if } a \in A_3 \\ 1/3 & \text{if } a \in A_4 \end{cases}$$

$$\omega_5'(a) = \begin{cases} 1/7 & \text{if } a \in A_1 \\ 1/5 & \text{if } a \in A_2 \\ 0.20 & \text{if } a \in A_3 \\ 1/3 & \text{if } a \in A_4 \end{cases} \qquad \omega_6'(a) = \omega_7'(a) = \begin{cases} 1/7 & \text{if } a \in A_1 \\ 0 & \text{if } a \in A_2 \\ 0.15 & \text{if } a \in A_3 \\ 0 & \text{if } a \in A_4, \end{cases}$$

where

$$\omega_1''(a) = \begin{cases} 1/7 & \text{if } a \in A_1 \\ 0.10 & \text{if } a \in A_2 \\ 0 & \text{if } a \in A_3 \\ 0 & \text{if } a \in A_4 \end{cases} \qquad \omega_2''(a) = \begin{cases} 1/7 & \text{if } a \in A_1 \\ 0.25 & \text{if } a \in A_2 \\ 0 & \text{if } a \in A_3 \\ 0 & \text{if } a \in A_4 \end{cases}$$

$$\omega_3''(a) = \begin{cases} 1/7 & \text{if } a \in A_1 \\ 0.15 & \text{if } a \in A_2 \\ 0.25 & \text{if } a \in A_3 \\ 1/3 & \text{if } a \in A_4 \end{cases} \qquad \omega_4''(a) = \begin{cases} 1/7 & \text{if } a \in A_1 \\ 0.25 & \text{if } a \in A_2 \\ 0.25 & \text{if } a \in A_3 \\ 1/3 & \text{if } a \in A_4 \end{cases}$$

$$\omega_5''(a) = \begin{cases} 1/7 & \text{if } a \in A_1 \\ 0.25 & \text{if } a \in A_2 \\ 0.20 & \text{if } a \in A_3 \\ 1/3 & \text{if } a \in A_4 \end{cases} \qquad \omega_6''(a) = \omega_7''(a) = \begin{cases} 1/7 & \text{if } a \in A_1 \\ 0 & \text{if } a \in A_2 \\ 0.15 & \text{if } a \in A_3 \\ 0 & \text{if } a \in A_4. \end{cases}$$

Figure 2 showcases a visual representation of the weights of F_Ω and $F_{\Omega'}$ on A_3. From the definition, $\omega_i'(\mathbf{a}) = \omega_i(\mathbf{a})$, for all $i \in \{1, \ldots, 7\}$ and $\mathbf{a} \in A_1 \cup A_2 \cup A_4$. When $\mathbf{a} \in A_3$, we obtain that

$$\omega_1(a) \leq \omega_1'(a), \qquad \omega_2(a) \leq \omega_2'(a), \qquad \omega_3(a) \leq \omega_3'(a), \qquad \omega_4(a) \leq \omega_4'(a),$$
$$\omega_5(a) \leq \omega_5'(a), \qquad \omega_6(a) \geq \omega_6'(a), \qquad \omega_7(a) \geq \omega_7'(a).$$

Using Proposition 3.3, we conclude that $F_\Omega \leq F_{\Omega'}$ in A_3, as intended.

Similarly, assume that the user requires the inputs on the region A_2 to yield lower outputs. To achieve this, we could consider the OFWA operator $F_{\Omega''} : [0,1]^7 \to [0,1]$ defined by the weights

$$\omega_1''(a) = \begin{cases} 1/7 & \text{if } a \in A_1 \\ 0.10 & \text{if } a \in A_2 \\ 0 & \text{if } a \in A_3 \\ 0 & \text{if } a \in A_4 \end{cases} \qquad \omega_2''(a) = \begin{cases} 1/7 & \text{if } a \in A_1 \\ 0.25 & \text{if } a \in A_2 \\ 0 & \text{if } a \in A_3 \\ 0 & \text{if } a \in A_4 \end{cases}$$

$$\omega_3''(a) = \begin{cases} 1/7 & \text{if } a \in A_1 \\ 0.15 & \text{if } a \in A_2 \\ 0.25 & \text{if } a \in A_3 \\ 1/3 & \text{if } a \in A_4 \end{cases} \qquad \omega_4''(a) = \begin{cases} 1/7 & \text{if } a \in A_1 \\ 0.25 & \text{if } a \in A_2 \\ 0.25 & \text{if } a \in A_3 \\ 1/3 & \text{if } a \in A_4 \end{cases}$$

$$\omega_5''(a) = \begin{cases} 1/7 & \text{if } a \in A_1 \\ 0.25 & \text{if } a \in A_2 \\ 0.20 & \text{if } a \in A_3 \\ 1/3 & \text{if } a \in A_4 \end{cases} \qquad \omega_6''(a) = \omega_7''(a) = \begin{cases} 1/7 & \text{if } a \in A_1 \\ 0 & \text{if } a \in A_2 \\ 0.15 & \text{if } a \in A_3 \\ 0 & \text{if } a \in A_4. \end{cases}$$

Comparing $F_{\Omega'}$ with $F_{\Omega''}$ in A_2, we can see that

$$\omega_1'(\mathbf{a}) \geq \omega_1''(\mathbf{a}), \quad \omega_2'(\mathbf{a}) \leq \omega_2''(\mathbf{a}), \quad \omega_3'(\mathbf{a}) \geq \omega_3''(\mathbf{a})$$

for all $\mathbf{a} \in A_2$. Therefore, we cannot use Proposition 3.3 to conclude which one is greater. However, simple calculations show that the operators satisfy the hypothesis of Proposition 3.1, thus we conclude that $F_{\Omega''} \leq F_{\Omega'}$ in A_2. A representation of these weights can be found in Figure 2.

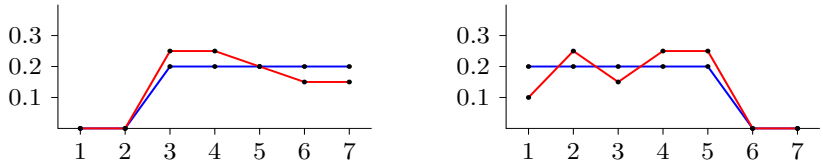


Figure 2. On the left: the weights ω_i (blue) and ω_i' (red) in the region A_3. On the right: the weights ω_i' (blue) and ω_i'' (red) in the region A_2.

5. Classes of OFWA operators

OFWA operators were introduced as a generalization of OWA operators, and indeed the latter are a subclass of the former. This section will consider several classes of aggregation operators [8, 28, 29, 33], some of them already extensions of OWA operators, and prove that they also are a subclass of OFWA operators.

One of the simplest examples of OWA operator is the arithmetic mean. Given the input values $(a_1, \dots, a_n) \in [0,1]^n$, their arithmetic mean is the value

$$F_{\text{am}}(a_1, \dots, a_n) = \frac{1}{n} \sum_{i=1}^{n} a_i.$$

This operator can be generalized by changing the weights, and the resulting operator is the weighted mean. Mainly, given a list of weights $P = (p_1, \dots, p_n)$ such that $\sum_{i=1}^{n} p_i = 1$, the weighted mean of the values $(a_1, \dots, a_n) \in [0,1]^n$ is

$$F_P(a_1, \dots, a_n) = \sum_{i=1}^{n} p_i a_i. \tag{3}$$

Comparing the previous expression with the definition for the OWA operator, they look similar. However, they have one important difference: The value corresponding to the weight p_i is not the i-th greatest input value but the value from the i-th source. Hence, they are different, in general, and coincide when the weights are the same, that is, if we are considering the arithmetic mean. Therefore, if we define the list of weights $\Omega_P = (\omega_1, \dots, \omega_n)$ as

$$\omega_i(a_1, \dots, a_n) = p_{\rho(i)}$$

for all $i \in \{1, \dots, n\}$ and $(a_1, \dots, a_n) \in [0,1]^n$, where $\rho\colon \{1, \dots, n\} \to \{1, \dots, n\}$ is a permutation on the index set such that $a_{\rho(n)} \leq \cdots \leq a_{\rho(1)}$, we obtain that the weighted mean F_P is an OFWA operator, more specifically, the OFWA operator F_{Ω_P}.

The OWA operator assigns weights based on the ordering of input values (from greatest to least), while the weighted mean assigns weights directly to the sources. These two approaches illustrate different perspectives in aggregation: The former emphasizes the values themselves, the latter the sources from which they originate. In [28, 29], the author introduced a new class of aggregation operator which merges both of these viewpoints, giving weights to each source and weights to each input value once ordered. This operator is introduced as below.

Definition 5.1. *Let $P = (p_1, \dots, p_n), W = (w_1, \dots, w_n) \in [0,1]^n$ be two lists of weights such that $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} w_i = 1$. The Weighted Ordered Weighted Averaging (WOWA) operator associated with the lists of weights P and W is the mapping $F_{P,W} : [0,1]^n \to [0,1]$ defined as*

$$F_{P,W}(a_1, \dots, a_n) = \sum_{i=1}^{n} \omega_i(a_1, \dots, a_n) a_{\rho(i)}$$

for all $(a_1, \ldots, a_n) \in [0,1]^n$*, where* $\rho : \{1, \ldots, n\} \to \{1, \ldots, n\}$ *is a permutation on the index set satisfying that* $a_{\rho(n)} \leq \cdots \leq a_{\rho(1)}$ *and the weight* ω_i *is defined as*

$$\omega_i(a_1, \ldots, a_n) = w^* \left(\sum_{j \leq i} p_{\rho(j)} \right) - w^* \left(\sum_{j < i} p_{\rho(j)} \right)$$

with w^* *an isotone function that interpolates the points*

$$(0,0), \quad (i/n, \sum_{j \leq i} w_j), \quad i \in \{1, \ldots, n\}.$$

The mapping w^* *is required to be a straight line when the points can be interpolated in this way.*

In the original definition given in [28], the dependence of w on the input values did not appear explicit in the definition but rather was given as:

$$F_{P,W}(a_1, \ldots, a_n) = \sum_{i=1}^{n} \omega_i a_{\rho(i).}$$

However, it is clear that the input values define the permutation ρ, and this permutation, together with the list of weights P, define the weight ω_i. Therefore, $\Omega = (\omega_1, \ldots, \omega_n)$ is a list of functional weights that depend on the input values, and thus we conclude that $F_{W,P}$ is the OFWA operator associated with the list of weights Ω. In [29], the author used a more general definition for the WOWA operator where the interpolating function w^*, which is defined only by the list of weights W, is replaced by an arbitrary fuzzy quantifier. This definition also give rise to OFWA operators through the same reasoning.

Now, we will compare OFWA and Generalized Ordered Weighted Averaging (GOWA) operators. There are several notions of mean that are used alongside the arithmetic and weighted mean, such as the geometric mean or the harmonic mean. All of these means can be seen as particular cases of the generalized weighted mean: Given a list of weights $W = (w_1, \ldots, w_n)$ and a parameter $\lambda \in (-\infty, \infty)$, the generalized mean of the values $(a_1, \ldots, a_n) \in [0,1]^n$ is

$$F_{\text{gwm}}(a_1, \ldots, a_n) = \left(\sum_{i=1}^{n} w_i a_i^{\lambda} \right)^{1/\lambda}.$$

This notion together with the OWA operator give rise to the following generalization of the OWA operator [33].

Definition 5.2. *Let* $W = (w_1, \ldots, w_n) \in [0,1]^n$ *be a list of weights such that* $\sum_{i=1}^{n} w_i = 1$ *and* $\lambda \in (-\infty, \infty)$ *be a parameter. The Generalized Ordered Weighted Averaging (GOWA) operator associated with* W *and* λ *is the mapping* $M_W^{\lambda} \colon [0,1]^n \to [0,1]$ *defined as*

$$M_W^{\lambda}(a_1, \ldots, a_n) = \left(\sum_{i=1}^{n} w_i a_{\rho(i)}^{\lambda} \right)^{1\lambda}$$

for all $(a_1, \ldots, a_n) \in [0,1]^n$*, where* $\rho \colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ *is a permutation on the index set satisfying that* $a_{\rho(n)} \leq \cdots \leq a_{\rho(1)}$.

The previous definition includes the original definition of the OWA operator for $\lambda = 1$, and the variation of the harmonic and quadratic means applying the reordering of the input values for $\lambda = -1$ and $\lambda = 2$, respectively. Moreover, even though the definition does not make sense for $\lambda = 0$, via limits it can be shown that the resulting operator is a variation of the geometric mean, that is,

$$M_W^0(a_1, \ldots, a_n) = \prod_{i=1}^{n} a_{\rho(i)}^{w_i}.$$

All of the examples up until now were clearly recognizable as OFWA operators and their weights were easy to define. However, Definition 5.2 does not due to the exponent. Therefore, we cannot find a natural list of weights Ω with a clear correspondence with the GOWA operators. In order to show that the GOWA operator is a particular instance of an OFWA operator, we use the fact that it is bounded [33], such that

$$\min\{a_1, \ldots, a_n\} \leq M_W^{\lambda}(a_1, \ldots, a_n) \leq \max\{a_1, \ldots, a_n\}$$

for all $(a_1, \ldots, a_n) \in [0,1]^n$. Using Theorem 2.1, we conclude that GOWA operator is a particular case of OFWA operator. Moreover, if we define the list of weights $\Omega_W^{\lambda} = (\omega_1, \ldots, \omega_n)$ as

$$\omega_1(a_1, ..., a_n) = \begin{cases} 1 & \text{if } a_1 = ... = a_n \\ \dfrac{M_W^{\lambda}(a_1, \ldots, a_n) - \min\{a_1, ..., a_n\}}{\max\{a_1, ..., a_n\} - \min\{a_1, \ldots, a_n\}} & \text{otherwise} \end{cases}$$

$$\omega_j(a_1, ..., a_n) = 0, \;\; j \in \{2, ..., n-1\}$$

$$\omega_n(a_1, ..., a_n) = \begin{cases} 0 & \text{if } a_1 = ... = a_n \\ \dfrac{\max\{a_1, ..., a_n\} - M_W^{\lambda}(a_1, ..., a_n)}{\max\{a_1, ..., a_n\} - \min a_1, ..., a_n} & \text{otherwise} \end{cases}$$

for all $(a_1, ..., a_n) \in [0,1]^n$, then $F_{\Omega_W^{\lambda}} = M_W^{\lambda}$. This same reasoning can be applied to showing that the monotone transformation of an OFWA operator is also an OFWA operator. The next result describes this fact.

Proposition 5.1. *Given an OFWA operator $F_\Omega : [0,1]^n \to [0,1]$ and a continuous strictly monotone function $g : [0,1] \to [0,1]$, the operator $F_\Omega^g : [0,1]^n \to [0,1]$ defined as*

$$F_\Omega^g(a_1, ..., a_n) = g^{-1}(F_\Omega(g(a_1), ..., g(a_n)))$$

is an OFWA operator.

Proof. First, we will show that the operator F_Ω^g is well defined. The function $g\colon [0,1] \to [0,1]$ is strictly monotone, thus the inverse $g^{-1}\colon [a,b] \to [0,1]$ exists, where $[a,b] = g([0,1])$ due to g being continuous. Moreover, since F_Ω is an OFWA operator, it is bounded, hence

$$F_\Omega(g(a_1), \ldots, g(a_n)) \in [\min\{g(a_1), \ldots, g(a_n)\}, \max\{g(a_1), \ldots, g(a_n)\}] \subseteq [a,b].$$

Therefore, we can apply the inverse to obtain the operator F_Ω^g.

Now, we will show that F_Ω^g is also bounded. Let $(a_1, \ldots, a_n) \in [0,1]^n$ and, without loss of generality, we will assume that $\min\{a_1, \ldots, a_n\} = a_1$, $\max\{a_1, \ldots, a_n\} = a_n$, and additionally that g is monotone increasing. Therefore, we have

$$\begin{aligned} g(a_1) = \min\{g(a_1), \ldots, g(a_n)\} &\leq F_\Omega^g(g(a_1), \ldots, g(a_n)) \\ &\leq \max\{g(a_1), \ldots, g(a_n)\} = g(a_n). \end{aligned}$$

Then, because g is also monotone increasing, we have

$$a_1 = g^{-1}g(a_1) \leq g^{-1}(F_\Omega^g(g(a_1), \ldots, g(a_n))) \leq g^{-1}g(a_n) = a_n.$$

Since the choice of values was arbitrary, we conclude that F_Ω^g is bounded. Using Theorem 2.1, we can conclude that F_Ω^g is also an OFWA operator. □

Another important aggregation operator is the Choquet integral (see [2, 10, 22, 30]). This notion is associated to a discrete fuzzy measure, defined below as in [7].

Definition 5.3. *Given $N = \{1, \ldots, n\}$, a discrete fuzzy measure is a function $m : 2^N \to [0,1]$ satisfying the following conditions:*

(1) *Increasing:* $m(X) \leq m(Y)$ *for all* $X \subseteq Y \subseteq N$.
(2) *Boundary conditions:* $m(\varnothing) = 0$ *and* $m(N) = 1$.

One of the characterizations of the notion of discrete Choquet integral [8, 23] as the follows.

Definition 5.4. *Given* $N = \{1, \ldots, n\}$*, the discrete Choquet integral with respect to the discrete fuzzy measure* $m : 2^N \to [0,1]$ *is the mapping* $C_m : [0,1]^n \to [0,1]$ *given by*

$$C_m(a_1, \ldots, a_n) = \sum_{i=1}^{n} a_{\rho(i)} \cdot \left(m(\{j \mid a_j \geq a_{\rho(i)}\}) - m(\{j \mid a_j \geq a_{\rho(i+1)}\}) \right),$$

where $(a_1, \ldots, a_n) \in [0,1]^n$, $\rho\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ *is a permutation, such that* $a_{\rho(1)} \leq \cdots \leq a_{\rho(n)}$*, and* $a_{\rho(n+1)} = \infty$ *by convention.*

In [6, 17, 21, 24], an equivalent definition is used:

$$C_m(a_1, \ldots, a_n) = \sum_{i=1}^{n} (a_{\rho(i)} - a_{\rho(i-1)}) \cdot m(\{j \mid a_j \geq a_{\rho(i)}\}),$$

where $a_{\rho(0)} = 0$ by convention. Nonetheless, we will use the first expression since it will be more direct for our purpose. To show that the Choquet integral C_m is an OFWA operator, we define the list of weights $\Omega^m = (\omega_1^m, \ldots, \omega_n^m)$, where

$$\omega_i^m(a_1, \ldots, a_n) = m(\{j \mid a_j \geq a_{\rho(i)}\}) - m(\{j \mid a_j \geq a_{\rho(i+1)}\})$$

for all $i \in \{1, \ldots, n\}$. In this way, the Choquet integral can be rewritten as

$$C_m(a_1, \ldots, a_n) = \sum_{i=1}^{n} \omega^m(a_1, \ldots, a_n)\ a_{\rho(i)}.$$

We will first show that the codomain of these weights lies within the unit interval. Clearly, $\omega_i^m(a_1, \ldots, a_n) \leq 1$ since the discrete fuzzy measure is greater than 0 and less than 1. Moreover, since $a_{\rho(i)} \leq a_{\rho(i+1)}$ we obtain that

$$\{j \mid a_j \geq a_{\rho(i)}\} \supseteq \{j \mid a_j \geq a_{\rho(i+1)}\}.$$

Therefore, due to the monotonicity of the discrete fuzzy measure,

$$m(\{j \mid a_j \geq a_{\rho(i)}\}) \geq m(\{j \mid a_j \geq a_{\rho(i+1)}\})$$

and thus, $\omega_i^m(a_1, \ldots, a_n) \geq 0$. It remains to show that the weights add up to 1. For any $(a_1, \ldots, a_n) \in [0,1]^n$, we have

$$\begin{aligned}\omega_1^m(a_1, \ldots, a_n) + \omega_2^m(a_1, \ldots, a_n) &= m(\{j \mid a_j \geq a_{\rho(1)}\}) - m(\{j \mid a_j \geq a_{\rho(2)}\}) \\ + m(\{j \mid a_j \geq a_{\rho(2)}\}) - m(\{j \mid a_j \geq a_{\rho(3)}\}) &= m(\{j \mid a_j \geq a_{\rho(1)}\}) - m(\{j \mid a_j \geq a_{\rho(3)}\}).\end{aligned}$$

Repeating the same procedure, the sum of the weights is

$$\sum_{i=1}^{n} \omega^m(a_1, \ldots, a_n) = m(\{j \mid a_j \geq a_{\rho(1)}\}) - m(\{j \mid a_j \geq a_{\rho(n+1)}\}).$$

The two sets considered in the above expression are

$$\begin{aligned}\{j \mid a_j \geq a_{\rho(1)}\} &= \{j \mid a_j \geq \min\{a_1, \ldots, a_n\}\} = \{1, \ldots, n\} = N, \\ \{j \mid a_j \geq a_{\rho(n+1)}\} &= \{j \mid a_j \geq \infty\} = \varnothing.\end{aligned}$$

Thus, due to the boundary conditions of the discrete fuzzy measure,

$$\sum_{i=1}^{n} \omega_i^m(a_1, \ldots, a_n) = m(N) - m(\varnothing) = 1 - 0 = 1.$$

We have showed that the list Ω^m is a list of functional weights as defined in Definition 2.2. Hence, the OFWA operator $F_{\Omega^m} : [0,1]^n \to [0,1]$ is

$$F_{\Omega^m}(a_1, \ldots, a_n) = \sum_{i=1}^{n} \omega_i^m(a_1, \ldots, a_n)\ a_{\rho(i)}.$$

Therefore, the Choquet integral C_m is the OFWA operator F_{Ω^m}. Moreover, given a continuous and strictly monotone function $g\colon [0,1] \to [0,1]$, the generalized Choquet integral is the operator $C_{m,g} : [0,1]^n \to [0,1]$ defined as

$$C_{m,g}(a_1, \ldots, a_n) = g^{-1}(C_m(g(a_1), \ldots, g(a_n))),$$

where $C_m : [0,1]^n \to [0,1]$ is the Choquet integral with respect to the discrete fuzzy measure $m : 2^N \to [0,1]$. Using Proposition 5.1, this operator is also an OFWA operator.

Due to their flexibility, OFWA operators encompass a wide range of common operators, like the ones presented in this section. Furthermore, the class of OFWA operators include any bounded operator, as proven in [12], which implies that operators as significant as the Sugeno integral can also be regarded as particular cases of OFWA operators. Nevertheless, the most important goal is to find simple and interpretable expressions for the associated weights of an arbitrary bounded operator, as we have done above with the Choquet integral. In the future, more operators will be studied.

6. Conclusions and future work

This work continues the study of ordered functional weighted averaging operators started in [5, 19]. Here, we focus on comparing different OFWA operators to determine which one yields higher or lower values within a certain region, and provide three sufficient conditions. All the results are illustrated through examples, including a practical one concerning a photovoltaic facility. Lastly, we study some families of aggregation operators, like the weighted ordered weighted averaging operator and the Choquet integral, and show that they are particular instances of OFWA operators.

In the future, we will investigate more sufficient conditions to ensure the usual monotonicity or other variants, such as directional monotonicity [16]. We will also explore methods for aggregating OFWA operators, extending the conservative and aggressive viewpoints described in [5]. Moreover, we will study different representations of OFWA operators and how the weights can be modified in order to better tailor the OFWA operator to the user's requirements given in natural language.

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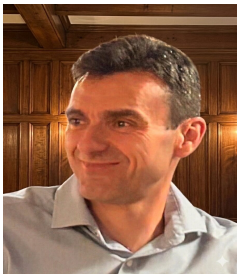
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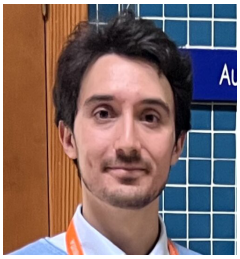
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Roberto G. Aragón - is a Professor at the University of Cádiz since 2021 in the Department of Mathematics, where he currently holds the position of Assistant Professor since 2023. His research interests include fuzzy sets, fuzzy rough sets, relational data analysis, formal concept analysis, and algebraic structures for soft computing.



Jesús Medina - is a Full Professor at the Department of Mathematics, University of Cádiz, Spain. He received the M.S. degree in mathematics in 1998 from the University of Granada, Spain, and the Ph.D. degree in mathematics in 2002 from the University of Málaga, Spain. His is interested in the research areas of fuzzy sets, fuzzy rough sets, fuzzy logic, residuated and multiadjoint logic programming, relational data analysis, and algebraic structures for soft computing.



Samuel Molina-Ruiz - is a Predoctoral Researcher at the Department of Mathematics, University of Cádiz, Spain. He received his Master's degree in Mathematics in 2024 from the University of Granada, Spain, and is currently pursuing his Ph.D. degree in Mathematics in the University of Cádiz. He is a member of the research group FQM406: Mathematics for computational intelligence systems (MCIS). His research interests include areas such as fuzzy sets, formal concept analysis, multi-adjoint frameworks, and aggregation operators.